Lecture 4
THE METHOD OF CALCULATION THE BAR ON RIGIDITY
Plan

1. Introduction.
2. Consideration of some typical examples.
3. Thin-walled pressure vessels.

4.1. Introduction.

In Fig. 4.1, determine the total elongation of an initially straight bar of length ℓ. Cross - sectional area A, and modulus of elasticity E if a tensile load F acts on the ends of the bar.
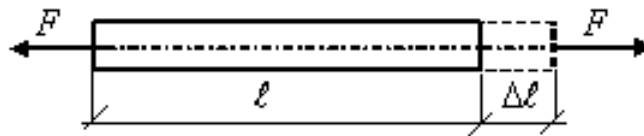

Fig. 4.1

The unit stress in the direction of the force F is merely the load divided by the cross-sectional area, that is:

$$\sigma = \frac{F}{A}.$$

Also the unit strain ε is given by the total elongation $\Delta\ell$ divided by the original length, i.e., $\varepsilon = \Delta\ell/\ell$. By definition the modulus of elasticity E is the ratio of σ to ε, that is:

$$E = \frac{\sigma}{\varepsilon} = \frac{F/A}{\Delta\ell/\ell} = \frac{F\ell}{\Delta\ell A}, \quad \text{or} \quad \Delta\ell = \frac{F\ell}{EA}. \tag{4.1}$$

Note that $\Delta\ell$ has the units of length, perhaps inches or meters.

Let us determine moving of free end of bar, which was considered last lecture (Fig. 3.2):

$$\Delta\ell = \Delta\ell_I + \Delta\ell_{II} + \Delta\ell_{III} + \Delta\ell_{IV} + \Delta\ell_V;$$

$$\Delta\ell_I = \frac{N_I \cdot \ell_I}{E \cdot A_1} = 0,$$

$$\Delta\ell_{II} = \frac{N_{II} \cdot \ell_{II}}{E \cdot A_1} = \frac{30 \cdot 10^3 \cdot 0,5 \cdot 10^3}{2 \cdot 10^5 \cdot 1,9 \cdot 10^2} = 0,394 \text{ mm,}$$

$$\Delta\ell_{III} = \frac{N_{III} \cdot \ell_{III}}{E \cdot A_2} = \frac{30 \cdot 10^3 \cdot 0,1 \cdot 10^3}{2 \cdot 10^5 \cdot 3,1 \cdot 10^2} = 0,0484 \text{ mm,}$$

$$\Delta\ell_{IV} = \frac{N_{IV} \cdot \ell_{IV}}{E \cdot A_2} = -\frac{8 \cdot 10^3 \cdot 0,4 \cdot 10^3}{2 \cdot 10^5 \cdot 3,1 \cdot 10^2} = -0,0516 \text{ mm,}$$

$$\Delta\ell_V = \frac{N_V \cdot \ell_V}{E \cdot A_2} = -\frac{50 \cdot 10^3 \cdot 0,2 \cdot 10^3}{2 \cdot 10^5 \cdot 3,1 \cdot 10^2} = -0,161 \text{ mm.}$$

Then: $\Delta\ell = 0,384 + 0,0484 - 0,0516 - 0,161 \cong 0,23$ mm.

Thus, the bar is lengthened on a 0,23 mm.

Consider a solid truncated conical bar of circular cross section tapers uniformly from a diameter d at its small end to D at the large end. The length of the bar is ℓ.

We shall determine the elongation due to an axial force F applied at each end. See Fig. 4.2.

The coordinate x describes the distance from the small end of a disc-like element of thickness dx. The radius of this small element is readily found by similar triangles:

$$r = \frac{d}{2} + \frac{x}{\ell}\left(\frac{D-d}{2}\right).$$

The elongation of this disc-like element may be found by applying the formula for extension due to axial loading:

$$\Delta\ell = \frac{F\ell}{EA}$$

the element, this expression becomes:

$$d\Delta\ell = \frac{F \cdot dx}{\pi\left(\dfrac{d}{2} + \dfrac{x}{\ell}\left(\dfrac{D-d}{2}\right)\right)^2 E}.$$
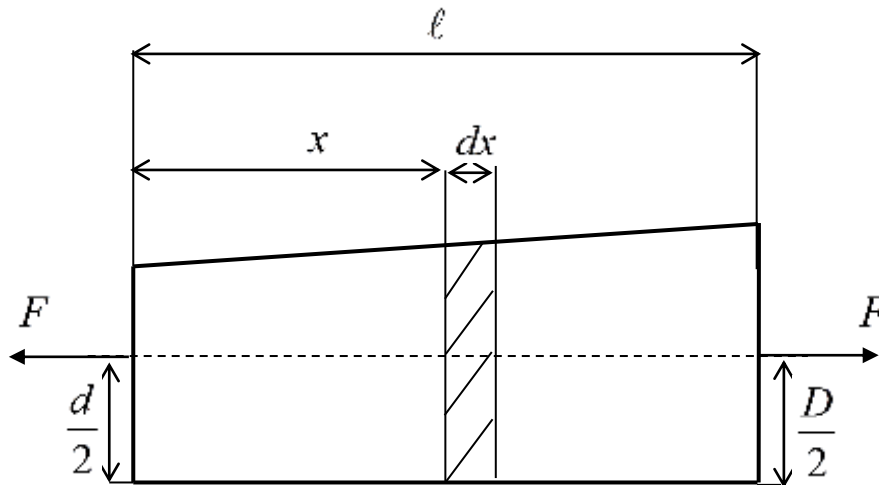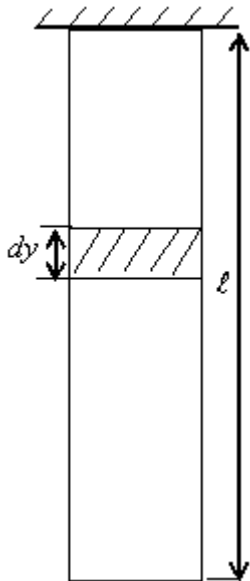

Fig. 4.2

The extension of the entire bar is obtained by summing the elongations of all such elements over the bar. This is of course done by integrating. If $\Delta\ell$ denotes the elongation of the entire bar:

$$\Delta\ell = \int_0^L d\Delta\ell = \int_0^L \frac{F \cdot dx}{\pi\left(\dfrac{d}{2} + \dfrac{x}{\ell}\left(\dfrac{D-d}{2}\right)\right)^2 E} = \frac{4FL}{\pi D d E}. \quad (4.2)$$

In Fig. 4.3, determine the total increase of length of a bar of constant cross section hanging vertically and subject to its own weight as the only load. The bar is initially straight.

The normal stress (tensile) over any horizontal cross section is caused by the weight of the material below that section. The elongation of the element of thickness by shown is:

Fig. 4.3

$$d\Delta\ell = \frac{A \cdot y \cdot \gamma}{A \cdot E} dy,$$

where A denotes the cross-sectional area of the bar and γ its specific weight (weight/unit volume). Integrating, the total elongation of the bar is:

$$\Delta\ell = \int_0^\ell \frac{A \cdot y \cdot \gamma}{A \cdot E} dy = \frac{A \cdot \gamma}{A \cdot E} \frac{\ell^2}{2} = \frac{A\ell}{2AE} \ell = \frac{W\ell}{2AE}, \qquad (4.3)$$

where W denotes the total weight of the bar. Note that the total elongation produced by the weight of the bar is equal to that produced by a load of half its weight applied at the end.

On beginning

4.2. Consideration of some typical examples.
We shall consider some typical examples.

Example 1.

A steel bar of cross section 500 mm^2 is acted upon by the forces shown in Fig. 4.4.

Determine the total elongation of the bar. For steel, consider E = 200 GPa.

The entire bar is in equilibrium, and hence all portions of it are also. The portion between A and B has a resultant force of 50 kN acting over every cross section and a free-body diagram of this 0,6 m length appears as in Fig.4.5.
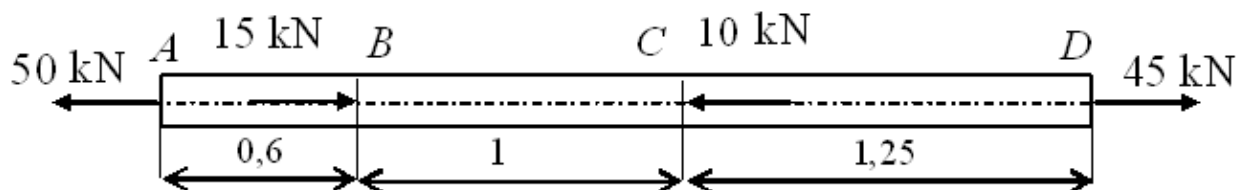

Fig. 4.4

The force at the right end of this segment must be 50 kN to maintain equilibrium with the applied load at A. The elongation of this portion is:

$$\Delta\ell_1 = \frac{50000 \cdot 0,6}{500 \cdot 10^{-6} \cdot 200 \cdot 10^9} = 0,0003 \text{ m.}$$
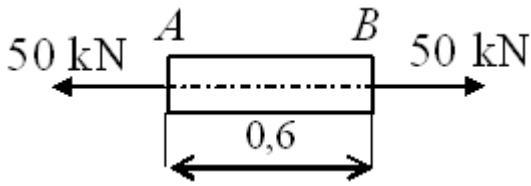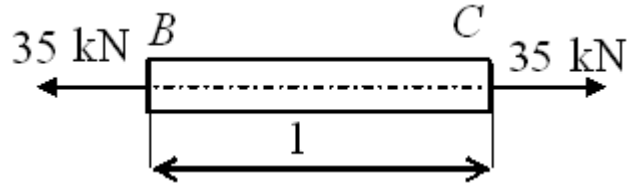
Fig. 4.5

Fig. 4.6

The force acting in the segment between B and C is found by considering the algebraic sum of the forces to the left of any section between B and C i.e., a resultant force of 35 kN acts to the left. so that a tensile force exists. The free-body diagram of the segment between B and C is shown in Fig. 4.6 and the elongation of it is:

$$\Delta \ell_2 = \frac{35000 \cdot 1}{500 \cdot 10^{-6} \cdot 200 \cdot 10^9} = 0,00035 \text{ m.}$$

Similarly, the force acting over any cross section between C and D must be 45 kN to maintain equilibrium with the applied load at D. The elongation of CD is:

$$\Delta \ell_2 = \frac{45000 \cdot 1,25}{500 \cdot 10^{-6} \cdot 200 \cdot 10^9} = 0,00056 \text{ m.}$$
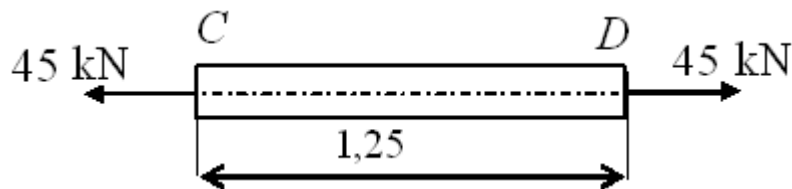
The total elongation is:

Fig. 4.7

$$\Delta \ell = \Delta \ell_1 + \Delta \ell_2 + \Delta \ell_3 = 0,00121 \text{ m} \qquad \text{or} \qquad \Delta \ell = 1,21 \text{ mm.}$$

Example 2.
In 1989, Jason, a research-type submersible with remote TV monitoring capabilities and weighing 35,200 N was lowered to a depth

of 646 m in an effort to send back to the attending surface vessel photographs of a sunken Roman ship offshore from Italy. The submersible was lowered at the end of a hollow steel cable having an area of $452 \cdot 10^{-6}$ m² and $E = 200$ GPa. The central core of the cable contained the fiber-optic system for transmittal of photographic images to the surface ship.

Determine the extension of the steel cable. Due to the small volume of the entire system buoyancy may be neglected, and the effect of the fiber-optic cable on the extension is also negligible. (Note: Jason was the system that took the first photographs of the sunken Titanic in 1986.)

The total cable extension is the sum of the extensions due to (4.1) the weight of Jason:

$$\Delta \ell_1 = \frac{F\ell}{EA} = \frac{35{,}2 \cdot 646}{452 \cdot 10^{-6} \cdot 200 \cdot 10^9} = 0{,}252 \text{ m,}$$

and from (4.3) we have for the weight of the steel cable:

$$\Delta \ell_2 = \frac{W\ell}{2AE},$$

where W is the weight of the cable, W may be found as the volume of the cable:

$$452 \cdot 10^{-6} \cdot 646 = 0{,}292 \text{ m}^3,$$

which must be multiplied by the weight of steel per unit volume which is 77 kN/m³. Thus, the cable weight is:

$$W = 0{,}292 \cdot 77 = 22{,}484 \text{ N,}$$

so that the elongation due to the weight of the cable is:

$$\Delta \ell_2 = \frac{22{,}484 \cdot 646}{2\left(452 \cdot 10^{-6} \cdot 200 \cdot 10^9\right)} = 0{,}080 \text{ m.}$$

The total elongation is the sum of the effects,

$$\Delta \ell = \Delta \ell_1 + \Delta \ell_2 = 0{,}252 + 0{,}080 = 0{,}332 \text{ m.} \textbf{On beginning}$$

4.3. Thin-walled pressure vessels

We examined various cases involving uniform normal stresses acting in bars. Another application of uniformly distributed normal stresses occurs in the approximate analysis of thin-walled pressure vessels, such as cylindrical spherical conical or toroidal shells subject to internal or external pressure from a gas or a liquid. In this we will treat only thin shells of revolution and restrict ourselves to axisymmetric deformations of these shells.
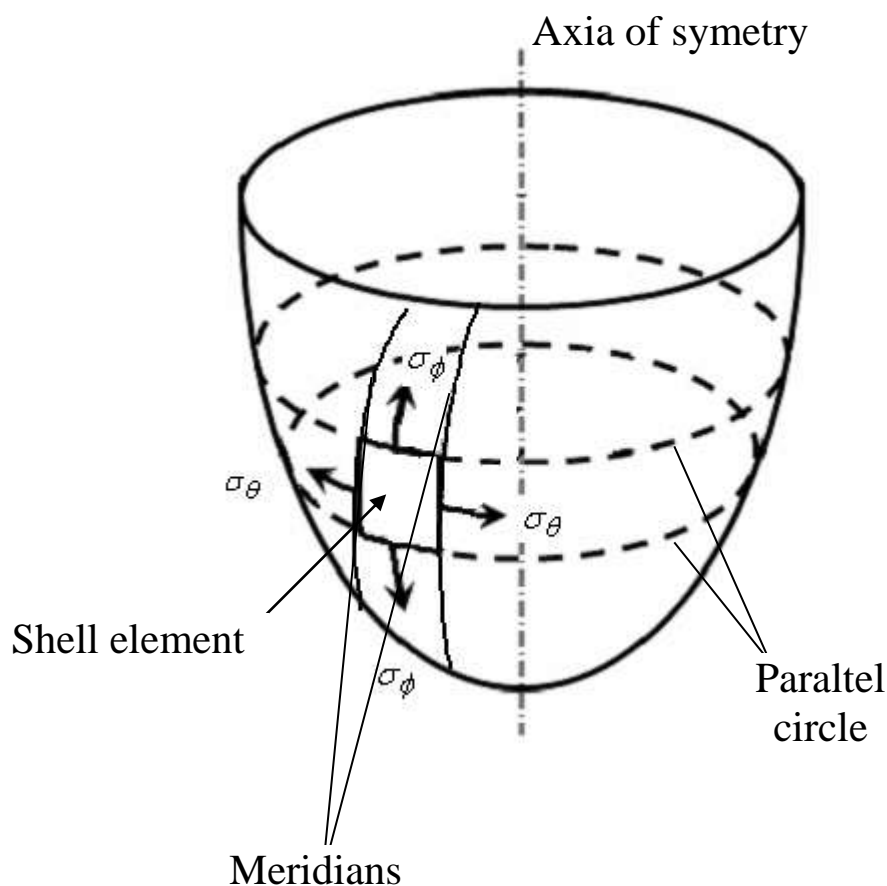

Fig. 4.8

The shell of revolution shown in Fig. 4.8 is formed by rotating a plane curve (the meridian) about an axis lying in the plane of the curve. The radius of curvature of the meridian is denoted by r_1 and this of course varies along the length of the meridian. This radius of curvature is defined by two lines perpendicular to the shell and passing through points B and C of Fig. 4.9 Another parameter, r_2, denotes the radius of curvature of the shell surface in a direction perpendicular to the meridian. This radius of curvature is defined by perpendiculars to

the shell through points *A* and *B* of Fig. 4.8. The centre of curvature corresponding to r_2 must lie on the axis of symmetry of the shell although the centre for r_1 in general does not lie there. An internal pressure *p* acting normal to the curved surface of the shell gives rise to meridional stresses σ_ϕ and hoop stresses σ_θ as indicated in the figure. These stresses are orthogonal to one another and act in the plane of the shell wall.

We have:

$$\frac{\sigma_\phi}{r_1} + \frac{\sigma_\theta}{r_2} = \frac{p}{h},$$

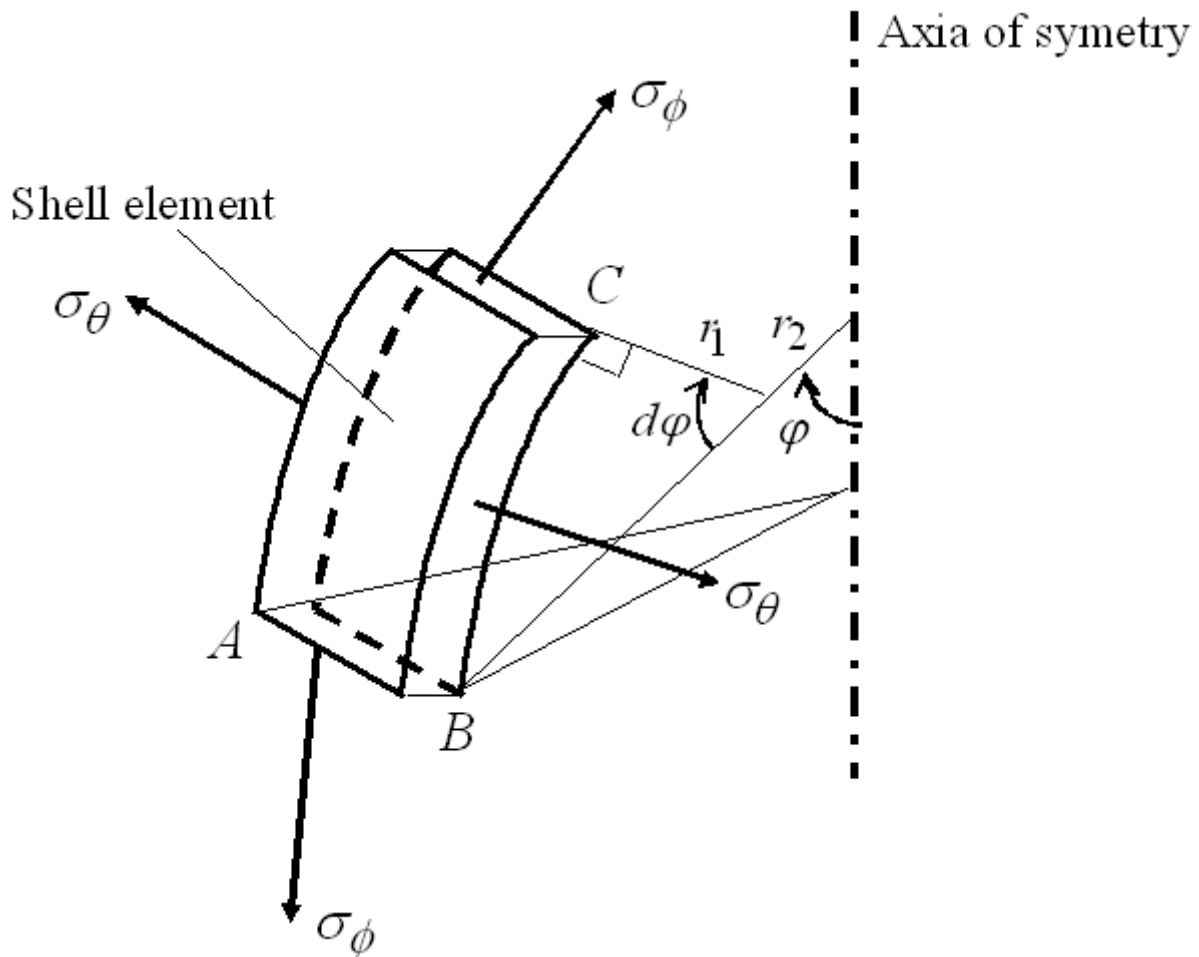where *h* denotes the shell thickness.

Fig. 4.9

A second equation may be obtained by consideration of the vertical equilibrium of the entire shell above some convenient parallel

circle.

The derivation of the above equation assumes that the stresses σ_ϕ and σ_θ are uniformly distributed over the wall thickness.

The ratio of the wall thickness to either radius of curvature should not exceed approximately 0.10. Also there must be no discontinuities in the structure. The simplified treatment presented here does not permit consideration of reinforcing rings on a cylindrical shell as shown in Fig. 4.10, nor does it give an accurate indication of the stresses and deformations in the vicinity of end closure plates on cylindrical pressure vessels. Even so, the treatment is satisfactory in many design problems.
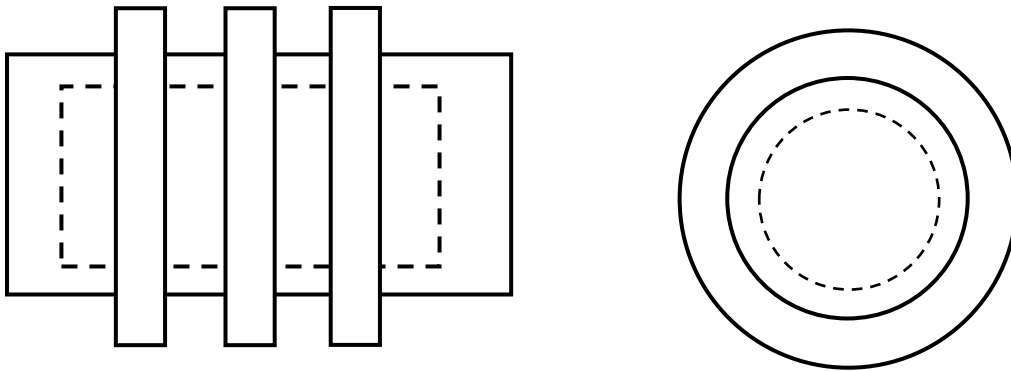

Fig. 4.10

The problems which follow are concerned with stresses arising from a uniform internal pressure acting on a thin shell of revolution. The formulas for the various stresses will be correct if the sense of the pressure is reversed, i.e., if external pressure acts on the container. However, it is to be noted that an additional consideration beyond the scope of this book, must then be taken into account. Not only must the stress distribution be investigated but another study of an entirely different nature must be carried out to determine the load at which the shell will buckle due to the compression. A buckling or instability failure may take place even though the peak stress is far below the maximum allowable working stress of the material.

Consider a thin-walled cylinder closed at both ends by cover plates and subject to a uniform internal pressure p. The wall thickness is h and the inner radius r. Neglecting the restraining effects of the end-plates, calculate the longitudinal (meridional) and circumferential

(hoop) normal stresses existing in the walls due to this loading.

To determine the circumferential stress σ_c let us consider a section of the cylinder of length ℓ to be removed from the vessel. The free-body diagram of half of this section appears as in Fig. 4.11. Note that the body has been cut in such a way that the originally internal effect σ_c now appears as an external force to this free body. Fig. 4.12 shows the forces ac1ing on a cross section.
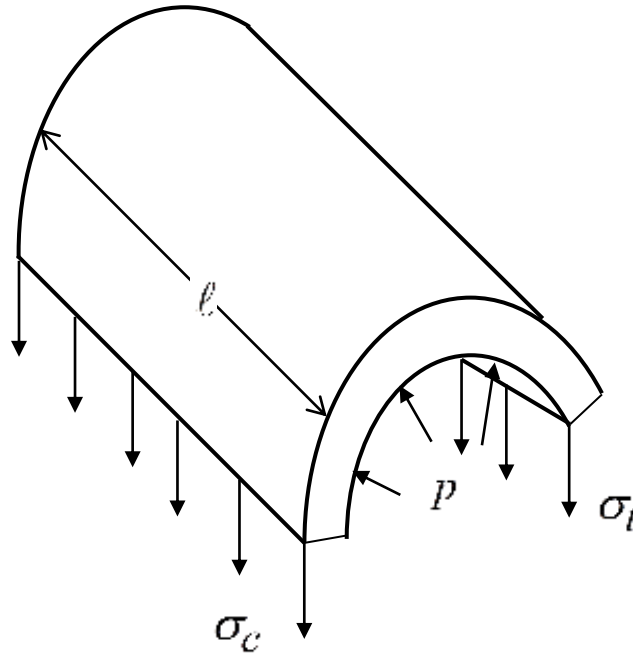

Fig. 4.11

The horizontal components of the radial pressures cancel one another by virtue of symmetry about the vertical centreline. In the vertical direction we have the equilibrium equation:

$$\Sigma F_{v_i} = -2\sigma_c h\ell + \int_0^\pi pr(d\theta)\ell \sin \theta = 0.$$

Integrating,

$$2\sigma_c h\ell = pr\ell[\cos \theta]_0^\pi,$$

or

$$\sigma_c = \frac{pr}{h}. \qquad (4.4)$$

Note that the resultant vertical force due to the pressure p could have been obtained by multiplying the pressure by the horizontal projected area upon which the pressure acts.
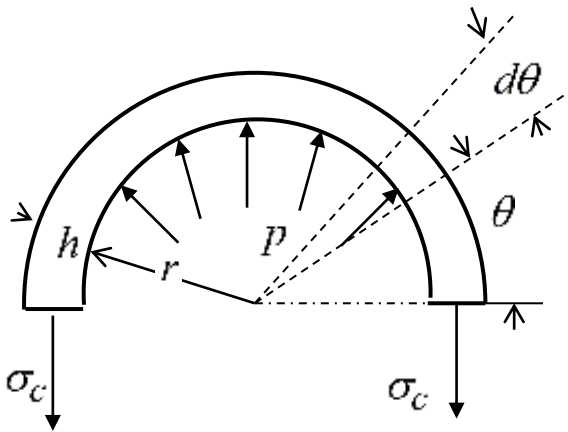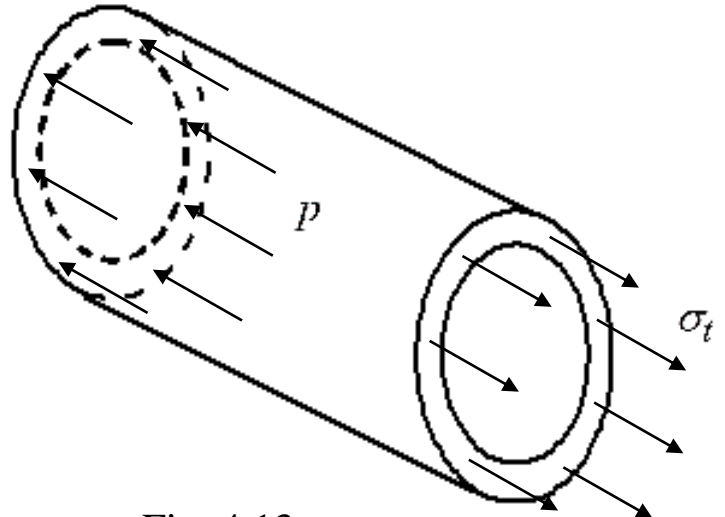

Fig. 4.12   Fig. 4.13

To determine the longitudinal stress σ_t consider a section to be passed through the cylinder normal to its geometric axis. The free-body diagram of the remaining portion of the cylinder is shown in Fig. 4.13. For equilibrium:

$$\Sigma F_{h_i} = -p\pi r^2 + 2\pi r h \sigma_t = 0,$$

or

$$\sigma_t = \frac{pr}{2h}. \qquad (4.5)$$

Consequently, the circumferential stress is twice the longitudinal stress. These rather simple expressions for stresses are not accurate in the immediate vicinity of the end closure plates.

Consider a closed thin-walled spherical shell subject to a uniform internal pressure p. The inside radius of the shell is r and its wall thickness is h. Derive an expression for the tensile stress existing in the wall.

For a free-body diagram, let us consider exactly half of the entire

sphere. This body is acted upon by the applied internal pressure p as well as the forces that the other half of the sphere, which has been removed, exerts upon the half under consideration. Because of the symmetry of loading and deformation, these forces may be represented by circumferential tensile stresses σ_c, as shown in Fig. 4.14.
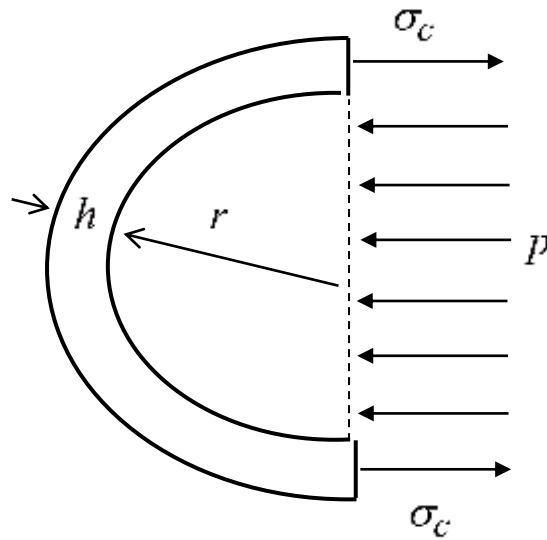

Fig. 4.14

This free-body diagram represents the forces acting on the hemisphere, the diagram showing only a projection of the hemisphere on a vertical plane. Actually the pressure p acts over the entire inside surface of the hemisphere and in a direction perpendicular to the surface at every point. It is permissible to consider the force exerted by this same pressure p upon the projection of this area which in this case is the vertical circular are denoted by a-a. This is possible because the hemisphere is symmetric about the horizontal axis and the vertical components of the pressure annul one another. Only the horizontal components produce the tensile stress σ_c. For equilibrium we have:

$$\Sigma F_{h_i} = -p\pi r^2 + 2\pi r h \sigma_c = 0,$$

or

$$\sigma_c = \frac{pr}{2h}.$$

From symmetry this circumferential stress is the same in all

directions at any point in the wall of the sphere.

Find the increase of volume of a thin-walled spherical shell subject to a uniform internal pressure p.

We know that the circumferential stress is constant through the shell thickness and is given by:

$$\sigma_c = \frac{pr}{2h}$$

in all directions at any point in the shell. From the two-dimensional form of Hooke's law, we have the circumferenlial strain as:

$$\varepsilon_c = \frac{1}{E}[\sigma_c - \mu\sigma_c] = \frac{pr}{2hE}[1 - \mu].$$

This strain is the change of length per unit length of the circumference of the sphere, so the increase of length of the circumference is:

$$(2\pi r) \cdot \frac{pr}{2hE}[1 - \mu].$$

The radius of the spherical shell subject to internal pressure p is now found by dividing the circumference of the pressurized shell by the factor 2π. Thus the final radius is:

$$\left[2\pi r + (2\pi r) \cdot \frac{pr}{2hE}[1 - \mu]\right] \bigg/ 2\pi, \tag{4.6}$$

or

$$r + \frac{pr^2}{2hE}[1 - \mu], \tag{4.7}$$

and the volume of the pressurized sphere is:

$$\frac{4}{3}\pi\left[r + \frac{pr^2}{2hE}[1 - \mu]\right]^3. \tag{4.8}$$

The desired increase of volume due to pressurization is found by subtracting from (4.7) the initial volume:

$$\Delta V = \frac{4}{3}\pi\left[r + \frac{pr^2}{2hE}[1-\mu]\right]^3 - \frac{4}{3}\pi r^3.$$

Expanding and dropping terms involving powers of (p/E), which is ordinarily of the order of 1/1000, we see that 1he increase of volume due to pressurization is:

$$\Delta V = \frac{2\pi pr^4}{hE}[1-\mu].$$

On beginning