Lecture 6
THE GEOMETRIC CHARACTERIZTIONS OF THE PLANE CROSS SECTIONS (continues)

Plan

1. The parallel - axis theorem for moment of inertia of a finite area.
2. Principal moments of inertia.
3. Solved problem.

6.1. The parallel - axis theorem for moment of inertia of a finite area.

This quantity has the dimension of a length to the fourth power, perhaps in^4 or m^4.

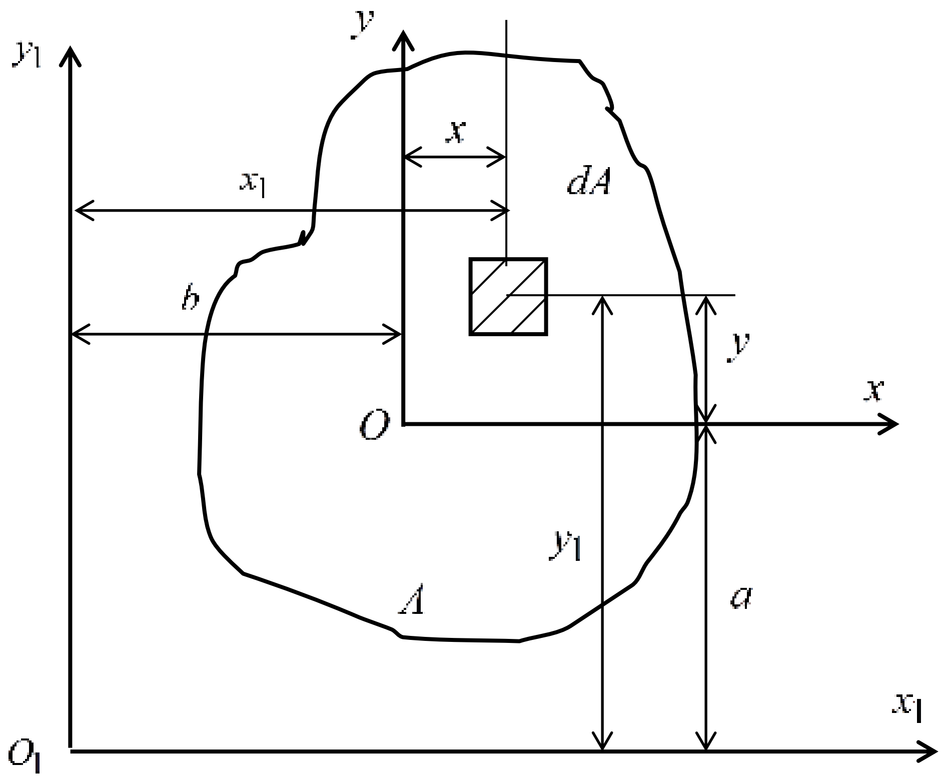

Fig. 6.1

For a plane area composed of n subareas A_i, each of whose moment of inertia is known about the x - and y - axes, the integral is replaced by a summation:

$$I_x = \sum_{i=1}^{n} I_{x_i}, \qquad I_y = \sum_{i=1}^{n} I_{y_i}. \tag{6.1}$$

The units of moment of inertia are the fourth power of a length, in^4 or m^4.

The parallel - axis theorem for moment of inertia of a finite area states that the moment of inertia of an area about any axis is equal to the moment of inertia about a parallel axis through the centroid of the area plus the product of the area and the square of the perpendicular distance between the two axes. For the area shown in Fig. 6.1, the axes x and y pass through the centroid of the plane area. The x_1 - and y_1 - axes are parallel axes located at distances a and b from the centroidal axes.

Let A denote the area of the figure, I_x and I_y the moments of inertia about the axes through the centroid, and I_{x_1} and I_{y_1} the moments of inertia about the x_1 - and y_1 - axes. Then we have:

$$I_{x_1} = I_x + a^2 A, \tag{6.2}$$

$$I_{y_1} = I_y + b^2 A. \tag{6.3}$$

Derive the parallel - axis theorem for moments of inertia of a plane area.

Let us consider the plane area A shown in Fig. 6.1. The axes x and y pass through its centroid, whose location is presumed to be known. The axes x and y are loca1ed at known distances a and b, respectively, from the axes through the centroid.

For the clement of area do the moment of inertia about the x_1 - axis is given by:

$$dI_{x_1} = (y + a)^2 dA.$$

For the entire area A the moment of inertia about the x_1 - axis is:

$$I_{x_1} = \int dI_{x_1} = \int_A (y+a)^2 dA = \int_A y^2 dA + \int_A a^2 dA + 2a \int_A y dA.$$

The second integral on the right is equal to:

$$a^2 \int_A dA = a^2 A,$$

because a is a constant. The third integral on the right is equal to:

$$2a \int_A y dA = 2a \cdot 0 = 0,$$

because the axis from which y is measured passes through the centroid of the area. The first in1egral on the right is equal to I_x i.e., the moment of inertia of the area about the horizontal axis through the centroid. Thus

$$I_{x_1} = I_x + a^2 A.$$

A similar considcra1ion in the other direction would show that:

$$I_{y_1} = I_y + b^2 A.$$

This is the parallel - axis theorem for plane areas. It is to be noted that one of the axes involved in each equa1ion must pass through the centroid of the area. In words, this may be stated as follows: the moment of inertia of an area with reference to an axis not through the centroid of the area is equal to the moment of inertia about a parallel axis through the centroid of the area plus the product of the same area and the square of the distance between the two axes.

The moment of inertia always has a positive value with a minimum value for axes through the centroid of the area in question.

If the moment of inertia of an area A about the x_1 - axis is denoted by I_{x_1}. Then the radius of gyration i_x is defined by:

$$i_x = \sqrt{\frac{I_x}{A}}. \tag{6.4}$$

Similarly, the radius of gyration with respect to the y - axis is given by:

$$i_y = \sqrt{\frac{I_y}{A}}. \qquad (6.5)$$

Since I is in units of length to the fourth power, and A is in units of length to the second power, then the radius of gyration has the units of length, say in or m. It is frequently useful for comparative purposes but has no physical significance.

The product of inertia of an element of area with respect to the x - and y - axes in the plane of the area is given by:

$$dI_{xy} = xydA,$$

where x and y are coordinates of the elemental area as shown in Fig. 6.1.

The product of inertia of a finite area with respect to the x - and y - axes in the plane of the area is given by the summation of the products of inertia about those same axes of all elements of area contained within the finite area. Thus:

$$I_{xy} = \int xydA. \qquad (6.6)$$

From this, it is evident that I_{xy} may be positive, negative, or zero. For a plane area composed of n subareas A, each of whose product of inertia is known with respect to specified x - and y - axes, the integral is replaced by the summation:

$$I_{xy} = \sum_{i=1}^{n} I_{x_i y_i}. \qquad (6.7)$$

The parallel - axis theorem for product of inertia of a finite area states that the product of inertia of an area with respect to the x - and y - axes is equal to the product of inertia about a set of parallel axes passing through the centroid of the area plus the product of the area and the two perpendicular distances from the centroid to the x_1 - and y_1 - axes. For the area, shown in Fig. 6.1, the axes x and y pass through the centroid of the plane area. The x - and y - axes are parallel axes located at distances x_1 and y_1 from the centroidal axes. Let A represent the area of the figure and I_{xy} be the product of inertia about the axes through the centroid. Then we have:

$$I_{x_1y_1} = I_{xy} + abA, \tag{6.8}$$

Let us derive the parallel - axis theorem for product of inertia of a plane area.

In Fig. 6.1 the axes x and y pass through the centroid of the area A. The axes x_1 and y_1 are located the known distances a and b, respectively, from the axes through the centroid.

For the element of area do the product of inertia with respect to the x_1 - and y_1 - axes is given by:

$$dI_{x_1y_1} = (y+a)(x+b)dxdy.$$

For the entire area the product of inertia with respect to the x_1 - and y_1 - axes becomes:

$$I_{x_1y_1} = \int dI_{x_1y_1} = \iint\limits_A (y+a)(x+b)dA = ab\int\limits_A dA + a\int\limits_A xdA + b\int\limits_A ydA + \int\limits_A xydA.$$

The first integral on the right side equals abA since a, and b are constants. The second and third integrals vanish because x and y are measured from the axes through the centroid of the area A. The fourth integral is equal to I_{xy}, that is, the product of inertia of the area with respect to axes through its centroid and parallel to the x_1 - and y_1 - axes. Thus, we have:

$$I_{x_1y_1} = I_{xy} + abA.$$

This is the parallel - axis theorem for product of inertia of a plane area. It is to be noted that the x - and y ·- axes must pass through the centroid of the area. Also, x_1 - and y_1 arc positive only when the x_1 - and y_1 - coordinates have the location relative to the xy system indicated in Fig. 6.1. Thus, care must be taken with regard to the algebraic signs of x and y.

On beginning

6.2. Principal moments of inertia.

Let us consider a plane area A and assume that I_x, I_y and I_{xy} are known. Determine the moments of inertia I_u and I_v as well as the product of inertia I_{uv} for the set of orthogonal axes u, v oriented as shown in Fig. 6.2. Determine also the maximum and minimum values of I_u.

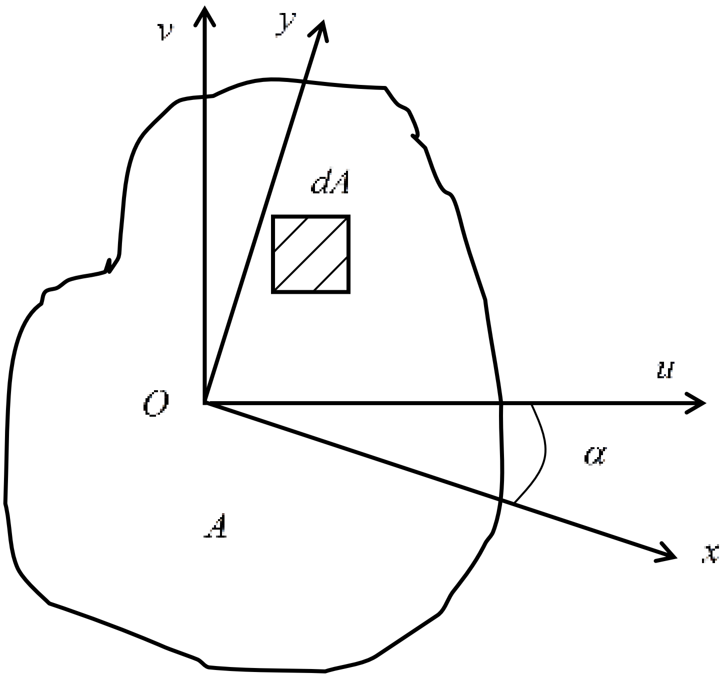

Fig. 6.2

The moment of inertia of the area with respect to the u - axis is:

$$I_u = \int_A (-x \cdot \sin\alpha + y \cdot \cos\alpha)^2 dA = \cos^2\alpha \int_A y^2 dA + \sin^2\alpha \int_A x^2 dA -$$

$$- \sin 2\alpha \int_A xy dA = I_x \cos^2\alpha + I_y \sin^2\alpha - I_{xy} \sin 2\alpha .$$

Or

$$I_u = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\alpha - I_{xy} \sin 2\alpha . \quad (6.9)$$

Analogously, I_v may be obtained from (6.9) by replacing α by $\alpha+\dfrac{\pi}{2}$ to yield:

$$I_v=\frac{I_x+I_y}{2}-\frac{I_x-I_y}{2}\cos 2\alpha+I_{xy}\sin 2\alpha . \qquad (6.10)$$

The value of α that renders I_u, maximum or minimum is found by setting the derivative of Eq. (6.9) with respect to α equal to zero. Thus, since I_u, I_v and I_{uv} are constants we have from (6.9):

$$\frac{dI_u}{d\alpha}=-\left(I_x-I_y\right)\sin 2\alpha-2I_{xy}\cos 2\alpha=0 .$$

Solving,

$$tg2\alpha=\frac{2I_{xy}}{\left(I_y-I_x\right)} . \qquad (6.11)$$

If now the values of 2α given by (6.11) are substituted into (6.8), we obtain:

$$I_{u_{\substack{\max\\ \min}}}=\frac{I_x+I_y}{2}\pm\frac{1}{2}\sqrt{(I_x-I_y)^2+4I_{xy}^2} , \qquad (6.12)$$

where the positive sign refers to Case I and the negative sign to Case II. These maximum and minimum values of moment of inertia correspond to axes defined by (6.11). The maximum and minimum values of moment of inertia are termed **principal moments of inertia** and the corresponding axes arc termed **principal axes**.

We may now determine I_{uv} from:

$$I_{uv}=\int_A\left(x\cos\alpha+y\sin\alpha\right)\left(y\cos\alpha-x\sin\alpha\right)dA=$$

$$=\cos^2\alpha\int_A xydA-\sin^2\alpha\int_A xydA+\cos\alpha\sin\alpha\int_A y^2dA+\cos\alpha\sin\alpha\int_A x^2dA=$$

$$= \frac{I_x - I_y}{2} \sin 2\alpha - I_{xy} \cos 2\alpha . \qquad (6.13)$$

From (6.13), I_{uv} vanishes if

$$tg2\alpha = \frac{2I_{xy}}{(I_y - I_x)},$$

which is identical to condition (6.11). Since (6.11) defined principal axes, it follows that the product of inertia vanishes for principal axes.

At any point in the plane of an area there exist two perpendicular axes about which the moments of inertia of the area are maximum and minimum for that point. These maximum and minimum values of moment of inertia are termed principal moments of inertia and are given by:

$$I_{\max} = \frac{I_x + I_y}{2} + \frac{1}{2}\sqrt{(I_x - I_y)^2 + 4I_{xy}^2},$$
$$I_{\min} = \frac{I_x + I_y}{2} - \frac{1}{2}\sqrt{(I_x - I_y)^2 + 4I_{xy}^2}. \qquad (6.14)$$

The pair of perpendicular axes through a selected point about which the moments of inertia of a plane area are maximum and minimum are termed principal axes.

The product of inertia vanishes if the axes are principal axes. Also, from the integral defining product of inertia of a finite area, it is evident that if either the x - axis, or the y - axis, or both, are axes of symmetry, the product of inertia vanishes. Thus, axes of symmetry are principal axes.

On beginning

5.3. Solved problem

Find basic geometrical characterizations for plane transversal section, represented on Fig. 6.3.

1. For finding of the centroid of the section and its geometrical characterizations will divide a section into three simple figures (Fig. 6.4): rectangle KDK_1K_2 and two triangles KK_2B and K_1DC. These triangles are conveniently considered together.

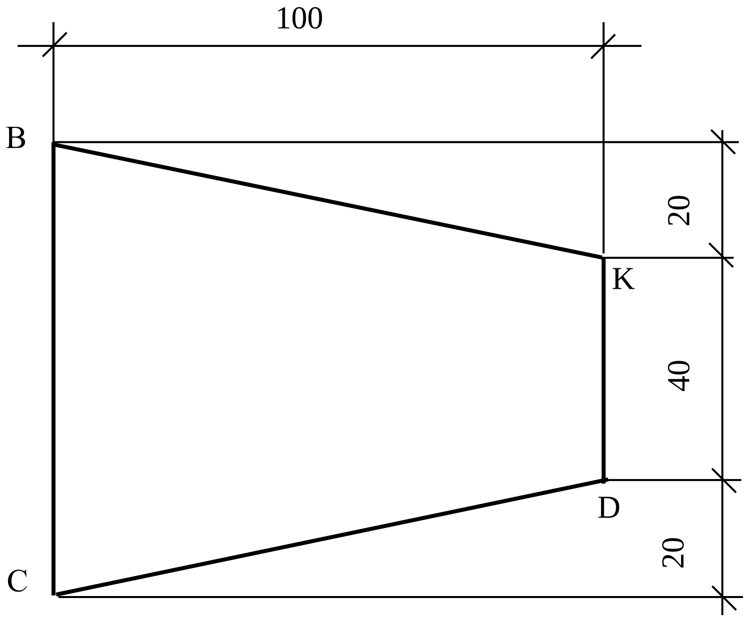

Fig. 6.3

2. Areas of constituents of the section are levels:

$$A_{KDK_1K_2} = 0,4 \cdot 1 = 0,4 \text{ см}^2 = 0,4 \text{ m}^2;$$

$$A_{KK_2B} = 0,5 \cdot 0,2 \cdot 1 = 0,1 \text{ см}^2 = 0,1 \text{ m}^2;$$

$$A_{K_1DC} = 0,5 \cdot 0,2 \cdot 1 = 0,1 \text{ см}^2 = 0,1 \text{ m}^2.$$

Then a general area a section is equal:

$$A = 2 \cdot 0,1 + 0,4 = 0,6 \text{ m}^2.$$

The section has a axis of symmetry x, and that is why determine the co-ordinate of centroid of x_c only ($y_c = 0$).

As $x_{c_1} = 0$, $x_{c_2} = x_{c_2} = -\frac{1}{6}$ m, then:

$$x_c = \frac{x_{c_1} \cdot A_1 + x_{c_2} \cdot (A_2 + A_3)}{A} = \frac{0,4 \cdot 0 + 0,2 \cdot \left(-\frac{1}{6}\right)}{0,6} = -\frac{1}{18} \text{ m} \approx -0,0556 \text{ m}.$$

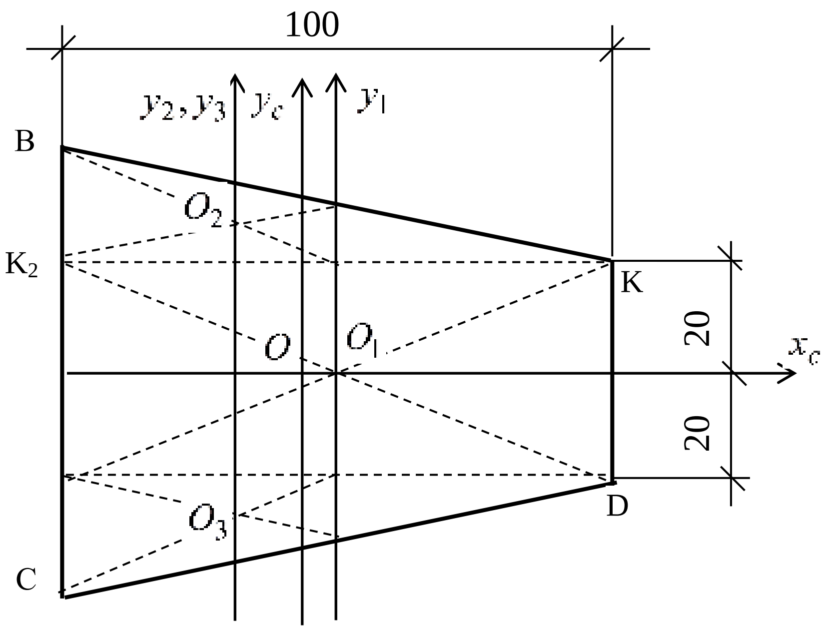

Fig. 6.4

2. At the calculation of moments of inertia will take into account that for triangles moments of inertia about the axes of x and y are determined by formulas:

$$I_x = \frac{bh^3}{36}, \quad I_y = \frac{b^3h}{36},$$

and the centrifugal moment of inertia is accordingly equal:

$$I_{xy} = -\frac{b^2h^2}{72}.$$

Then moments of inertia about central axes of section determined by formulas:

$$I_{x_c} = \sum_{i=1}^{3}\left(I_{x_{c_i}} + y_{c_i}^2 \cdot A_i\right) = \left(\frac{1\cdot 0{,}4^3}{12}\right) + 2\left(\frac{1\cdot 0{,}2^3}{36} + \left(0{,}2 + \frac{0{,}2}{3}\right)^2 \cdot 0{,}1\right) =$$
$$= 0{,}005333 + 0{,}014667 = 0{,}02 \text{ m}^4;$$

$$I_{y_c} = \sum_{i=1}^{3}\left(I_{y_{c_i}} + x_{c_i}^2 \cdot A_i\right) = \left(\frac{1^3\cdot 0{,}4}{12} + \left(-\frac{1}{18}\right)^2 \cdot 0{,}4\right) +$$

$$+2\left(\frac{1^3\cdot 0{,}2}{36}+\left(\frac{2}{18}\right)^2\cdot 0{,}1\right)=0{,}03456+0{,}01358=0{,}04814\ \text{m}^4.$$

As this section has an axis of symmetry x_C, the centrifugal moment of inertia about it is equal to the zero. And it means that axes x_C and y_C are principal central axes of given section. Accordingly, moments of inertia I_{x_C} and I_{y_C} are principal central moments of inertia for given transversal section

On beginning