Lecture 16
DOUBLE - INTEGRATION METHOD

Plan

1. General theory.
2. About the double – integration method.
3. The example of application.

16.1. General theory.

The deformation of a beam is most easily expressed in terms of the deflection of the beam from its original unloaded position. The deflection is measured from the original neutral surface to the

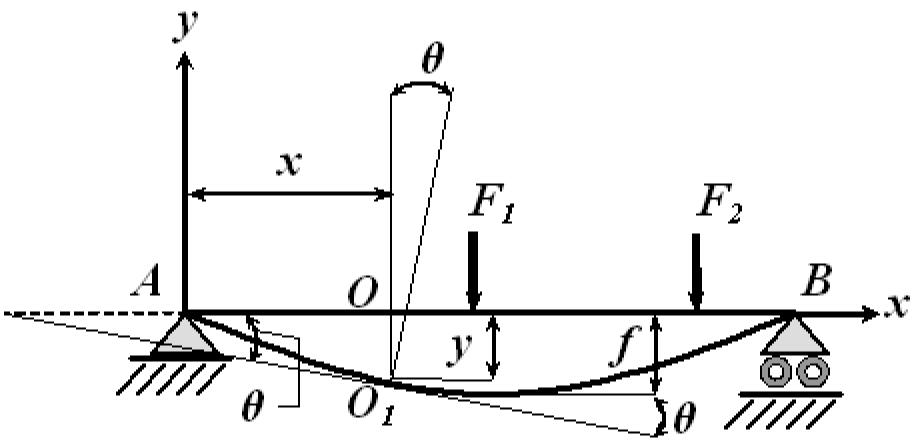

Fig. 16.1

neutral surface of the deformed beam. The configuration assumed by the deformed neutral surface is known as the elastic curve of the beam. Figure 16.1 represents the beam in the deformed configuration it has assumed under the action of the load.

The displacement f is defined as the deflection of the beam. Often it will be necessary to determine the deflection y for every value of x along the beam. This relation may be written in the form of an equation which is frequently called the equation of the deflection curve (or elastic curve) of the beam.

Specifications for the design of beams frequently impose limitations upon the deflections as well as the stresses. For example, in many building codes the maximum allowable deflection of a beam is not to exceed $\frac{1}{300}$ of the length of the beam. Components of aircraft usually are designed so that deflections do not exceed some reassigned value, else the aerodynamic characteristics may be altered. Thus, a well-designed beam must not only be able to carry the loads to which it will be subjected but it must not undergo undesirably large deflections. Also, the evaluation of reactions of statically indeterminate beams involves the use of various deformation relationships.

Numerous methods are available for the determination of beam deflections. The most commonly used are the following: double - integration method and elastic energy methods.

On beginning

16.2. Double - integration method. The differential equation of the deflection curve of the bent beam is:

$$EI\frac{d^2y}{dx^2} = M_x, \tag{16.1}$$

where x and y are the coordinates shown in Fig. 16.10. That is, y is the deflection of the beam. In the equation E denotes the modulus of elasticity of the beam and I represents the moment of inertia of the beam cross section about the neutral axis, which passes through the centroid of the cross section. Also, M_x represents the bending moment at the distance x from one end of the beam. Usually, M_x will be a function of x and it will be necessary to integrate (16.1) twice to obtain an algebraic equation expressing the deflection of y as a function of x.

Let us derive the equation (16.1). Obtain the differential equation of the deflection curve of a beam loaded by lateral forces.

From (16.9) we have the relationship:

$$M = \frac{EI_x}{\rho}. \tag{16.2}$$

In this expression M denotes the bending moment acting at a particular cross section of the beam, ρ the radius of curvature to the neutral surface of the beam at this same section, E the modulus of elasticity, and I_x the moment of the cross - sectional area about the neutral axis passing through the centroid of the cross section. In this book we will usually be concerned with those beams for which E and I_x are constant along the entire length of the beam, but in general both M and ρ will be functions of x.

Equation (16.2) may be written in the form:

$$\frac{1}{\rho} = \frac{M}{EI_x}. \tag{16.3}$$

where the left side of Eq. (16.3) represents the curvature of the neutral surface of the beam. Since M will vary along the length of the beam, the deflection curve will be of variable curvature.

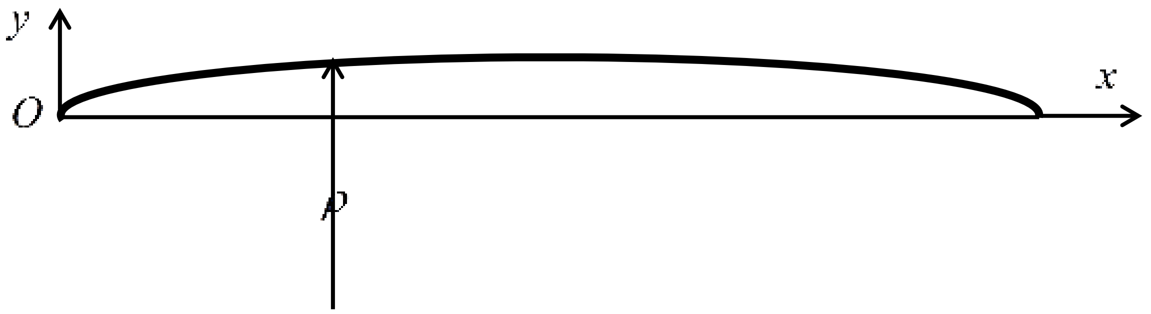

Fig. 16.2

Let the heavy line in Fig. 16.2 represent the deformed neutral surface of the bent beam. Originally the beam coincided with the x - axis prior to loading and the coordinate system that is usually found to be most convenient is shown in the sketch. The deflection y is taken to be positive in the upward direction: hence for the particular beam shown, all deflections are negative.

An expression for the curvature at any point along the curve representing the deformed beam is readily available from differential calculus. The exact formula for curvature is:

$$\frac{1}{\rho} = \frac{\dfrac{d^2y}{dx^2}}{\left(1+\left(\dfrac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}. \tag{16.4}$$

In this expression $\frac{dy}{dx}$ represents the slope of the curve at any point; and for small beam deflections this quantity and in particular its square are small in comparison to unity and may reasonably be neglected. This assumption of small deflections simplifies the expression for curvature into:

$$\frac{1}{\rho} \approx \frac{d^2 y}{dx^2} . \tag{16.5}$$

Hence for small deflections, (16.3) becomes:

$$\frac{d^2 y}{dx^2} = \frac{M}{EI_x} .$$

This is the differential equation of the deflection curve of a beam loaded by lateral forces. In honor of its co discoverers, it is called the Euler-Bernoulli equation of bending of a beam. In any problem it is necessary to integrate this equation to obtain an algebraic relationship between the deflection y and the coordinate x along the length of the beam.

On beginning

16.3. The example of application.

Determine the deflection at every point or the cantilever beam subject to the single concentrated force P, as shown in Fig. 16.3.

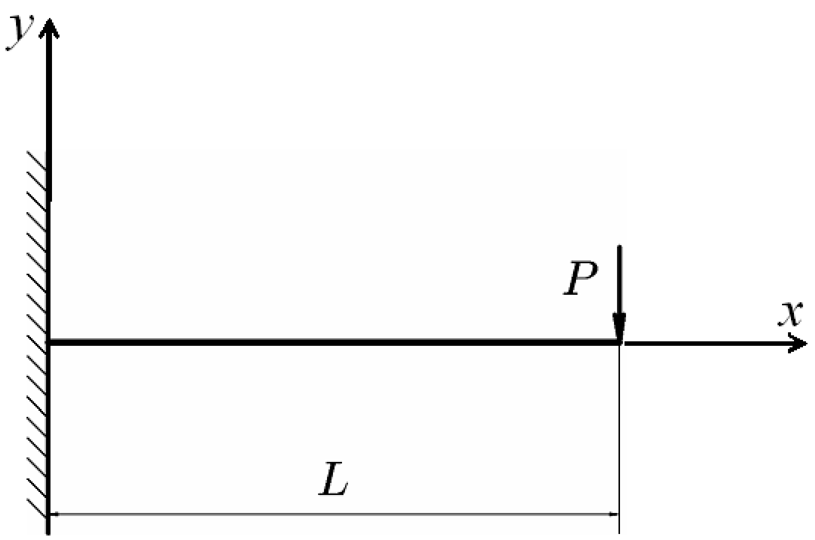

Fig. 16.3

The x - y coordinate system shown is introduced, where the x - axis coincides with the original unbent position of the beam. The deformed beam has the appearance indicated by the heavy line. It is first necessary to find the reactions exerted by the supporting wall upon the bar, and these are easily found from statics to be a vertical force reaction P and a moment PL, as shown in Fig. 16.4.

The bending moment at any cross section a distance x from the wall is given by the sum of the moments of these two reactions about an axis through this section. Evidently the upward force P produces a positive bending moment Px, and the couple PL. Hence the bending moment M at the section x is:

$$M_x = -PL + P \cdot x.$$

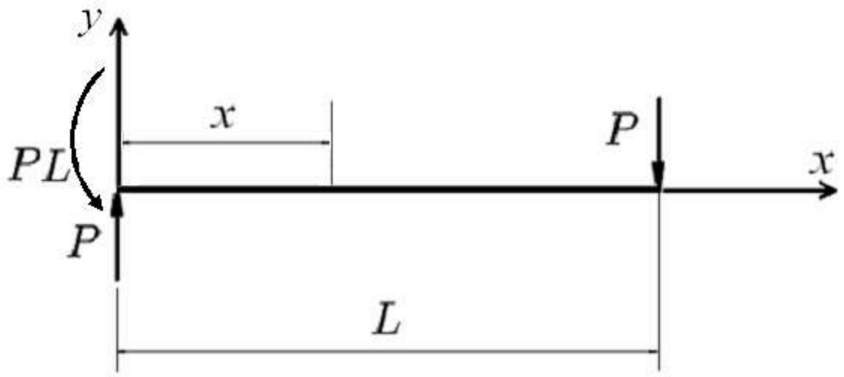

Fig. 16.4

The differential equation of the bent beam is:

$$EI_x \frac{d^2 y}{dx^2} = M_x,$$

where E denotes the modulus of elasticity of the material and I_x represents the moment of inertia of the cross section about the neutral axis. Substituting,

$$EI_x \frac{d^2 y}{dx^2} = -PL + P \cdot x. \qquad (16.6)$$

This equation is readily integrated once to yield:

$$EI_x \frac{dy}{dx} = -PLx + \frac{P \cdot x^2}{2} + C, \qquad (16.7)$$

which represents the equation of the slope, where C denotes a constant of integration.

This constant may be evaluated by use or the condition that the slope $\frac{dy}{dx}$ of the beam at the wall is zero since the beam is rigidly clamped there. Thus:

$$\frac{dy}{dx}\Big|_{x=0} = 0.$$

Equation (16.7) is true for al1 values of x and y, and if the condition $x = 0$ is substituted we obtain:

$$0 = 0 + 0 + C,$$

or

$$C = 0.$$

Next integration of (16.7) yields:

$$EI_x y = -\frac{PLx^2}{2} + \frac{P \cdot x^3}{6} + D, \qquad (16.8)$$

where D is a second constant of integration. Again, the condition at the supporting wall will determine this constant. There, at $x = 0$, the deflection y is zero since the bar is rigidly clamped. Substituting:

$$y\big|_{x=0} = 0$$

in Eq. (16.8). we find:

$$0 = 0 + 0 + D,$$

or:

$$D = 0.$$

Thus, Eqs. (16.7) and (16.8) with $C = D = 0$ give the slope $\frac{dy}{dx}$ and deflection y at any point x in the beam. The deflection is a maximum at the right end of the beam $x = L$, under the load P, and from Eq. (16.8):

$$EI_x y_{max} = -\frac{PL^3}{3}, \qquad (16.9)$$

where the negative value denotes that this point оп the deflection curve lies below the x - axis. If only the magnitude of the maximum deflection at $x = L$ is desired, it is usually denoted by δ_{max} and we have:

$$\delta_{max} = -\frac{PL^3}{3EI_x}, \qquad (16.10)$$

Equation (16.1) is the basic differential equation that governs the elastic deflection of all beams irrespective of the type of applied loading.

The quantities E and I appearing in (16.1) are, of course, positive. Thus, from this equation, if M_x is positive for a certain value of x, then $\frac{d^2y}{dx^2}$ is also positive. With the above sign convention for bending moments, it is necessary to consider the coordinate x along the length of the beam to be positive to the right and the deflection y to be positive upward. With these algebraic signs the integration of (16.1) may be carried out to yield the deflection y as a function of x, with the understanding that upward beam deflections are positive and downward deflections negative.

In the derivation of (16.1) it is assumed that deflections caused by shearing action are negligible compared to those caused by bending action. Also, it is assumed that the deflections are small compared to the cross - sectional dimensions of the beam and that all portions of the beam are acting in the elastic range. Equation (16.1) is derived on the basis of the beam being straight prior to the application of loads. Beams with slight deviations from straightness prior to loading may be treated by modifying.

On beginning