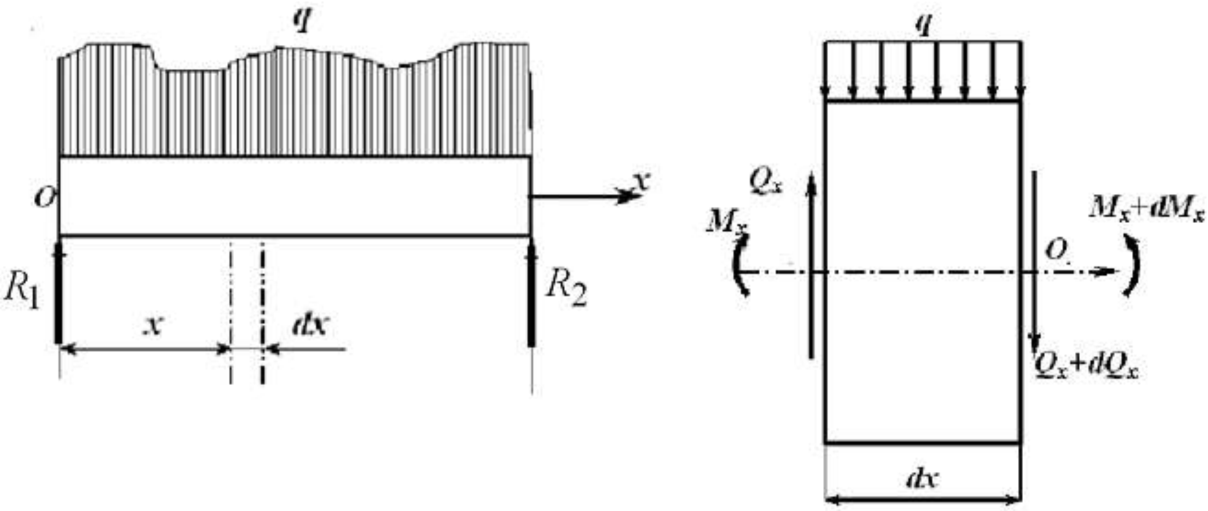
Lecture 12
STRESSES IN BEAMS

Plan

1. Shear and moment equation.
2. Normal sresses in beams.
3. The strength – weight ratio under torsion.

12.1. Shear and moment equation.

Usually, it is convenient to introduce a coordinate system along the beam, with the origin at one end of the beam. It will be desirable to know the shearing force and bending moment at all sections along the beam and for this purpose two equations are written, one specifying the shearing force Q as a function of the distance, say x, from one end of the beam, the other giving the bending moment M as a function of x.



a b

Fig. 12.1

The plots of these equations for Q and M are known as shearing force and bending moment diagrams, respectively. In these plots the abscissas (horizontals) indicate the position of the section along the beam and the ordinates (verticals) represent the values of the shearing force and bending moment, respectively. Thus, these diagrams represent graphically the variation of shearing force and bending moment at any section along the length of the bar. From these plots it is quite easy to determine the maximum value of each of these quantities.

Let us derive relationships between load intensity, shearing force and bending moment at any point in a beam.

Let us consider a beam subject to any type of transverse load of the general form shown in fig. 12.1, a. Simple supports are illustrated but the following consideration holds for all types of beams. We will isolate from the beam the clement of length dx shown and draw a free - body diagram of it. The shearing force Q acts on the left side of the clement, and in passing through the distance dx the shearing force will in general change slightly to an amount $Q+dQ$. The bending moment M acts on the left side of the element and it changes to $M+dM$ on the right side. Since dx is extremely small, the applied load may be taken as uniform over the top of the beam and equal to q. The free - body diagram of this element thus appears as in Fig. 12.1, b. For equilibrium of moments about O, we have:

$$\sum M_O = M-(M+dM)+Q\cdot dx+q\cdot dx\cdot\frac{dx}{2}=0,$$

or

$$dM = Q\cdot dx+\frac{1}{2}q\cdot(dx)^2.$$

Since, the last term consists of the product of two differentials; it is negligible compared with the other forms involving only one differential. Hence

$$dM = Q\cdot dx,$$

or

$$Q=\frac{dM}{dx}.$$

Thus, the shearing force is equal to the rate of change of the bending moment with respect to x.

This equation will prove to be of considerable value in drawing shearing force and bending moment diagrams for the more complicated types of loading. For example, from this equation it is evident that if the shearing force is positive at a certain section of the beam then the slope of the bending moment diagram is also positive at that point. Also, it demonstrates that an abrupt change in shear, corresponding to a concentrated force. is accompanied by an abrupt change in the slope of the bending moment diagram.

Further, at those points where the shear is zero, the slope of the bending moment diagram is zero. At these points where the tangent to the moment diagram is horizontal, the moment may have a maximum or minimum value. This follows from the usual calculus technique of obtaining maximum or minimum values of a function by equating the first derivative of the function to zero. Thus in Fig. 12.2 if the curves shown represent portions of a bending moment diagram then critical values may occur at points A and B.

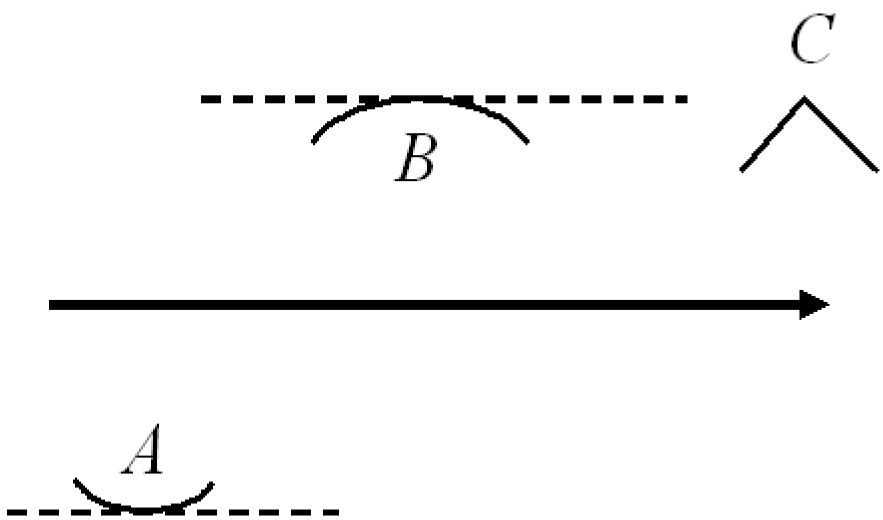


Fig. 12.2

To establish the direction of concavity at a point such as A or B , we may form the second derivative of M with respect to x, that is, $\frac{d^2M}{dx^2}$. If the value of this second derivative is positive, then the moment diagram is concave upward, as at A, and the moment assumes a minimum value. If the second derivative is negative the moment diagram is concave downward, as at B, and the moment assumes a maximum value.

However, it is to be carefully noted that the calculus method of obtaining critical values by use of the first derivative does not indicate possible maximum values at a cusp-like point in the moment diagram, if one occurs, such as that shown at C. If such a point is present, the moment there must be determined numerically and then compared to other values that are possibly critical.

Lastly, for vertical equilibrium of the element we have:

$$Q-(Q+dQ)+q\cdot dx=0,$$

or

$$q=\frac{dQ}{dx}.$$

This relation will be of value in establishing shearing force diagrams.

On beginning

12.2. Normal sresses in beams

Either forces or couples that lie in a plane containing the longitudinal axis of the beam may act upon the member. The forces are understood to act perpendicular to the longitudinal axis, and the plane containing the forces is assumed to be a plane of symmetry of the beam.

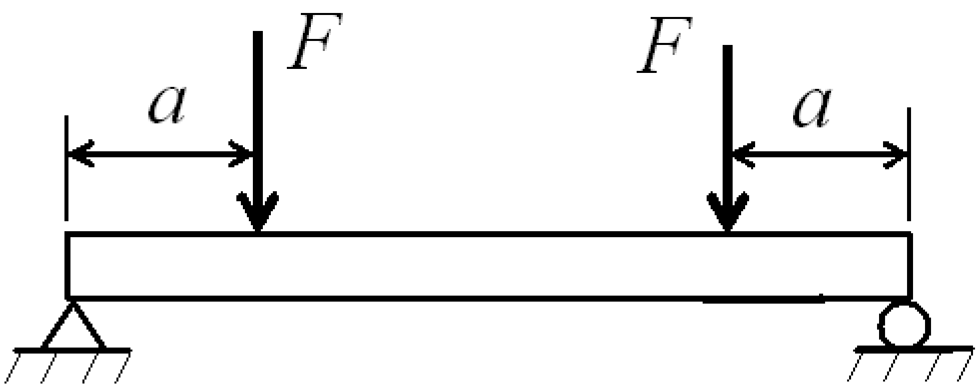


Fig. 12.3

If couples are applied to the ends of the beam and no forces act on the bar, then the bending is termed pure bending. For example, in Fig. 12.3 the portion of the beam between the two downward forces is subject to pure bending. Bending produced by forces that do not form couples is called ordinary bending. A beam subject to pure bending has only normal stresses with no shearing stresses set up in it; a beam subject to ordinary bending has both normal and shearing stresses acting within it.

It is convenient to imagine a beam to be composed of an infinite number of thin longitudinal rods or fibers. Each longitudinal fiber is assumed to act independently of every other fiber, i.e., there are no lateral pressures or shearing stresses between the fibers. The beam of Fig. 12.3, for example, will deflect downward and the fibers in the lower part of the beam undergo extension, while those in the upper part are shortened. These changes in the lengths of the fibers set up stresses in the fibers. Those that are extended have tensile stresses acting on the fibers in the direction of the longitudinal axis of the beam, while those that are shortened are subject to compressive stresses.

There always exists one surface in the beam containing fiЬers that do not undergo any extension or compression, and thus are not subject to any tensile or compressive stress. This surface is called the neutral surface of the beam.

The intersection of the neutral surface with any cross section of the beam perpendicular to its longitudinal axis is called the neutral axis. All fibers on one side of the neutral axis are in a state of tension, while those on the opposite side are in compression.

The algebraic sum of the moments of the external forces to one side of any cross section of the beam about an axis through that section is called the bending moment at that section.

The following remark apply only if all fibcrs in the beam are acting within the elastic range of action of the material.

Let us derive an expression for the relationship between the bending moment acting at any section in a beam and the bending stress at any point in this same sect ion. Assume Hooke's law holds.

The beam shown in Fig. 12.4, a is loaded by the two couples M and consequently is in static equilibrium. Since the bending moment has the same value at all points along the bar, the beam is said to be in a condition of pure bending. To determine the distribution of bending stress in the beam, let us cut the beam by a plane passing through it in a direction perpendicular to the geometric axis of the bar. In this manner the forces under investigation become external to the new body formed, even though they were internal effects with regard to the original uncut body.

The free - body diagram of the portion of the beam to the left of this cutting plane now appears as in Fig. 4.12, b. Evidently a moment M must act over the cross section cut by the plane so that the left portion of the beam will be in static equilibrium. The moment M acting on the cut section represents the effect of the right portion of the beam on the left portion. Since the right portion has been removed, it must be replaced by its effect on the left portion and this effect is represented by the moment M. This moment is the resultant of the moments of forces acting perpendicular to the cut cross section and in the plane of the page. It is now necessary to make certain assumptions in order to determine the nature of the variation of these forces over the cross section.

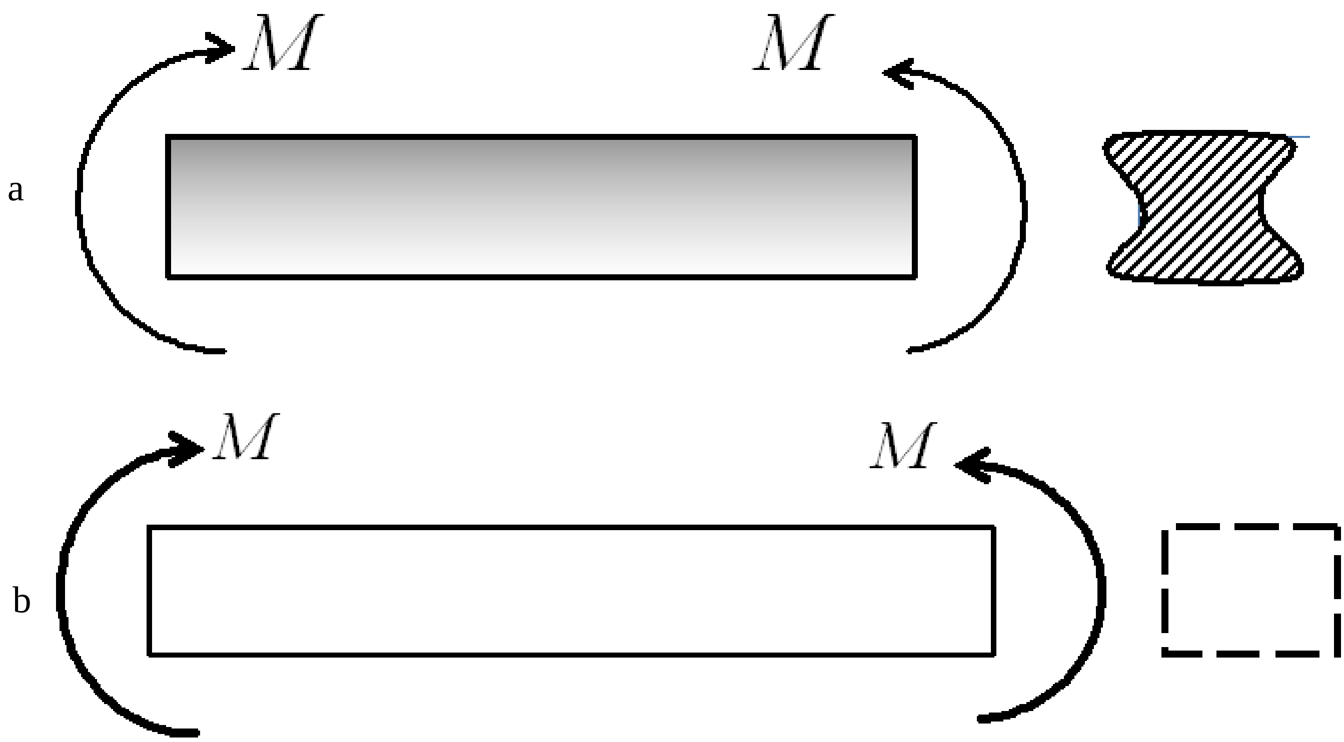


Fig. 12. 4

It is convenient to consider the beam to be composed of an infinite number of thin longitudinal rods or fibers. It is assumed that every longitudinal fiber acts independently of every other fibcr; that is, there arc no lateral pressures or shearing stresses between adjacent fibers. Thus each fiber is subject only to axial tension or compression. Further, it is assumed that a plane section of the beam norma1 to its axis before loads are applied remains plane and normal to the axis after loading. Finally, it is assumed that the material follows Hooke's law and that the module of elasticity in tension and compression are equal.

Let us next consider two adjacent cross sections a -a and b- b marked on the side of the beam, as shown in Fig. 12.5. Prior to loading, these sections arc parallel to each other. After the applied moments have acted on the beam, these sections are still planes but they have rotated with respect to each other to the positions shown, where O represents the centre of curvature of the beam. Evidently, the fibers on the upper surface of the beam are in a state of compression, while those on the lower surface have been extended

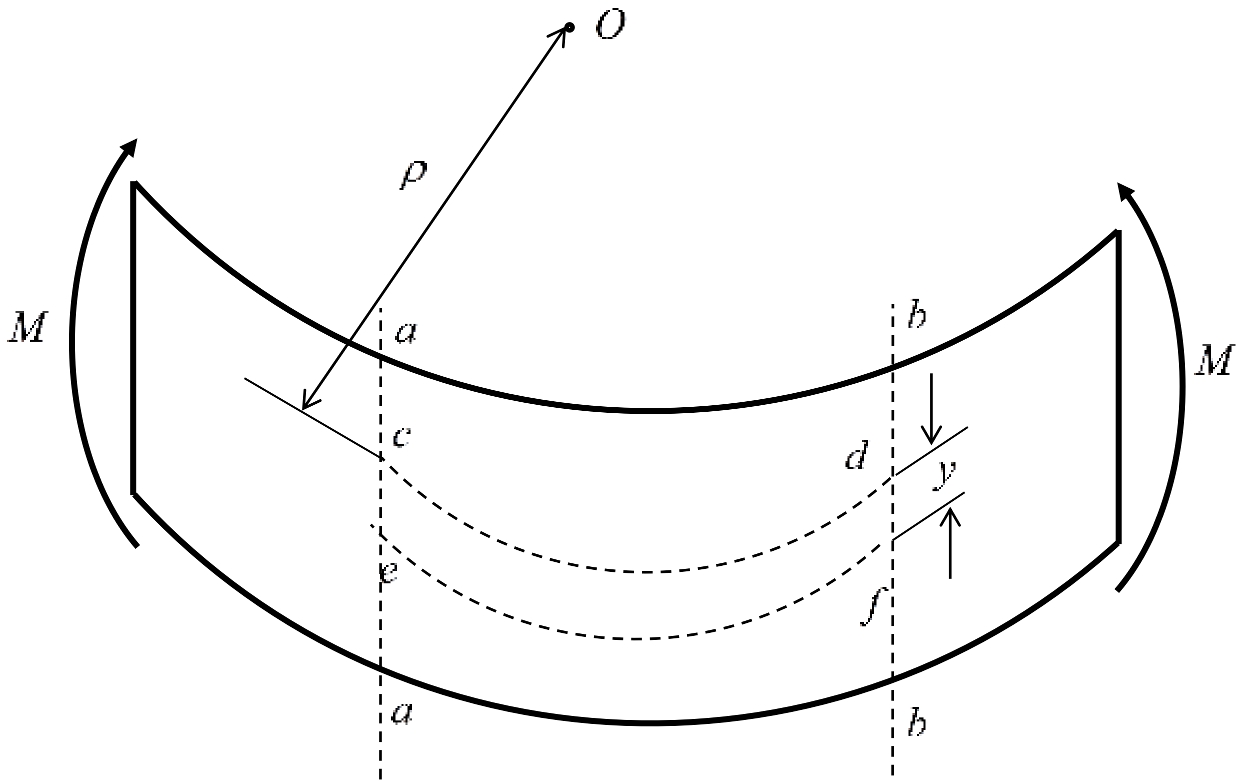


Fig. 12.5

slightly and arc thus in tension. The line *ed* is the trace of the surface in which the fibers do not undergo any strain during bending and this surface is called the neutral surface, and its intersection with any cross section is called the neutral axis. The elongation of the longitudinal fiber at a distance y (measured positive downward) may be found by drawing line *de* parallel to a - a. If ρ denotes the radius of curvature of the bent beam, then from the similar triangles *eOd* and *edf* we find the strain of this fiber to be:

$$\varepsilon = \frac{ef}{cd} = \frac{de}{cO} = \frac{y}{\rho}. \tag{12.1}$$

Thus, the strains of the longitudinal fibers are proportional to the distance y from the neutral axis.

Since Hooke's law holds, and therefore:

$$E = \frac{\sigma}{\varepsilon},$$

or

$$\sigma = E\varepsilon,$$

it immediately follows that the stresses existing in the longitudinal fibers are proportional to the distance y from the neutral axis, or

$$\sigma = E\frac{y}{\rho}. \tag{12.2}$$

Let us consider a beam of rectangular cross section, although the derivation actually holds for any cross section which has a longitudinal plane of symmetry. In this case, these longitudinal, or bending, stresses appear as in Fig. 12.5.

Let da represent an element of area of the cross section at a distance y from the neutral axis. The stress acting on da is given by the above expression and consequently the force on this element is the product of the stress and the area da, that is:

$$dF = E\frac{y}{\rho}da. \tag{12.3}$$

However, the resultant longitudinal force acting over the cross section is zero (for the case of pure bending) and this condition may be expressed by the summation of all forces dF over the cross section. This is done by integration:

$$\int E\frac{y}{\rho}da = \frac{E}{\rho}\int y da = 0. \tag{12.4}$$

Evidently, $\int yda = 0$. However, this integral represents the first moment of the area of the cross section with respect to the neutral axis, since y is measured from that axis. But, from Chap. 3 we may write:

$$\int yda = y_C A,$$

where y_C is the distance from the neutral axis to the centroid of the cross - sectional area. From this, $y_C A = 0$; and since A is not zero, then $y_C = 0$. Thus, the neutral axis always passes through the centroid of the cross section, provided Hooke's law holds.

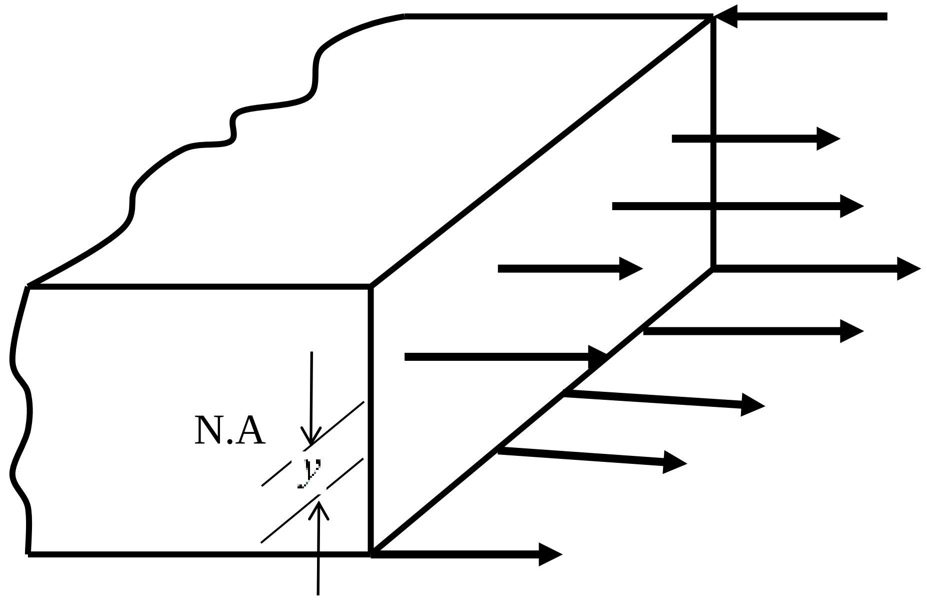


Fig. 12.6

The moment of the elemental force dF about the neutral axis is given by:

$$dM = y \cdot dF = y \cdot \left(E \frac{y}{\rho} da \right). \tag{12.5}$$

The resultant of the moments of a1l such elemental forces summed over the entire cross section must be equal to the bending moment M acting at that section and thus we may write:

$$M = \int E \frac{y^2}{\rho} da . \qquad (12.6)$$

But $I_x = \int y^2 da$ and thus we have:

$$M = \frac{EI_x}{\rho} . \qquad (12.7)$$

It is to be carefully noted that this moment of inertia of the cross - sectional area is computed with respect to the axis through the centroid of the cross section. But previously we had:

$$\sigma = E \frac{y}{\rho} . \qquad (12.8)$$

Eliminating ρ from these last two equations, we obtain:

$$\sigma = \frac{M}{I_x} y , \qquad (12.9)$$

This formula gives the so-called bending or flexural stresses in the beam. In it, M is the bending moment at any section, I_x the moment of inertia of the cross - sectional area about an axis through the centroid of the cross section, and y the distance from the neutral axis (which passes through the centroid) to the fiber on which the stress σ acts.

When the beam action is entirely elastic the neutral axis passes through the centroid of the cross section. Hence, the moment of inertia I_x appearing in the above equation for normal stress is the moment of inertia of the cross - sectional area about an axis through the centroid of the cross section of the beam.

At the outer fibers of the beam the value of the coordinate y is frequently denoted by the symbol c. In that case the maximum normal stresses are given by:

$$\sigma = \frac{M}{I_x / c} . \qquad (12.10)$$

The ration I_x/c is called the section modulus and is usually denoted by the symbol W. The units are in^3 or m^3. The maximum bending stresses may then be represented as:

$$\sigma = \frac{M}{W}. \qquad (12.11)$$

This form is convenient because values of W are available in handbooks for a wide range of standard structural steel shapes.

On beginning

12.3. Shearing stress in beams.

In the derivation of the above expression for normal stresses it is assumed that a plane section of the beam normal to its longitudinal axis prior to loading remains plane after the forces and couples have been applied. Further, it is assumed that the beam is initially straight and of uniform cross section and that the module of elasticity in tension and compression are equal. Again, it is to be emphasized that no fibers of the beam are stressed beyond the proportional limit.

The algebraic sum of all the vertical forces to one side of any cross section of the beam is called the ***shearing force*** at that section. This concept was discussed.

For any beam subject to a shearing force Q (expressed in pounds) at a certain cross section, both vertical and horizontal shearing stresses τ are set up. The magnitudes of the vertical shearing stresses at any cross section are such that these stresses have the shearing force Q as a resultant. In the cross section of the beam shown in Fig. 12.5, the vertical plane of symmetry contains the applied forces and the neutral axis passes through the centroid of the section. The coordinate y is measured from the neutral axis. The moment of inertia of the entire cross - sectional area about the neutral axis is denoted I_x. The shearing stress on all fibers a distance y0 from the neutral axis is given by the formula:

$$\tau = \frac{Q}{bI_x} \int_{y_0}^{c} y\,da, \qquad (12.12)$$

where b denotes the width of the beam at the location where the shearing stress is being calculated.

Let us consider an element of length dx cut from a beam as shown in Fig. 12.5. We shall denote the bending moment at the left side of the element by M and that at the right side by $M + dM$, since in general the bending moment changes slightly as we move from one section to an adjacent section of the beam. If y is measured upward from the neutral axis, then the bending stress at the left section a - a is given by (12.10).

This stress distribution is illustrated above. Similarly, the bending stress at the right section b - b is:

$$\sigma^* = \frac{M + dM}{I_x} y .$$

Let us now consider the equilibrium of the shaded element $acdb$. The force acting on an area da of the face ac is merely the product of the intensity of the force and the area, thus:

$$\sigma \cdot da = \frac{My}{I_x} da .$$

The sum of all such forces over the left face ac is found by integration to be:

$$\int_{y_0}^{c} \frac{My}{I_x} da .$$

Like wise, the sum of all normal forces over the right face bd is given by:

$$\int_{y_0}^{c} \frac{(M + dM)y}{I_x} da .$$

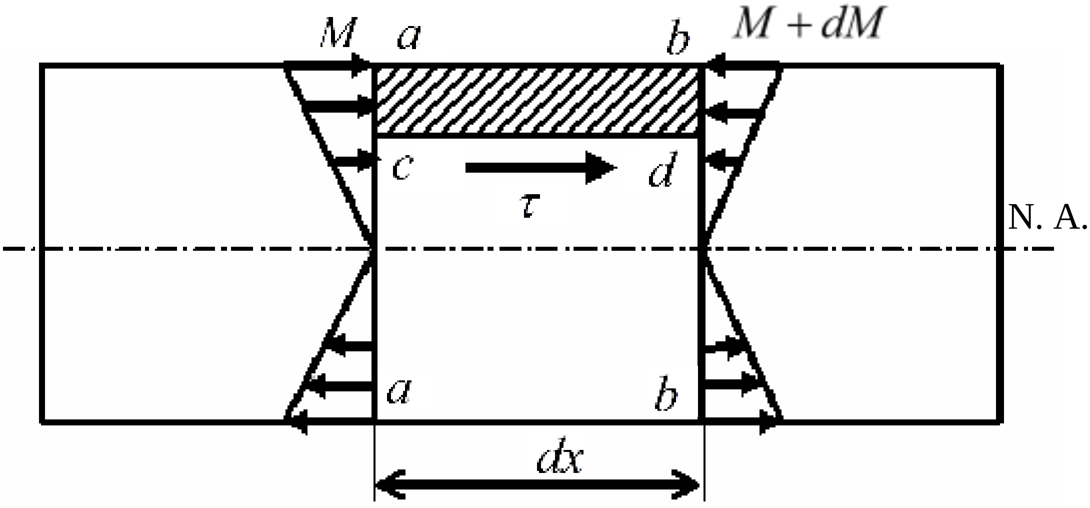


Fig. 12.7

Evidently, since these two integrals are unequal, some additional horizontal force must act on the shaded element to maintain equilibrium. Since the top face ab is assumed to be free of any externally applied horizontal forces, then the only remaining possibility is that there exists a horizontal shearing force along the lower face cd. This represents the action of the lower portion of the beam on the shaded element. Let us denote the shearing stress along this face by τ as shown. Also, let b denote the width of the beam at the position where τ acts. Then the horizontal shearing force along the face cd is $\tau b \ dx$. For equilibrium of the element $acdb$ we have:

$$\tau = \frac{1}{I_x b}\frac{dM}{dx}\int_{y_0}^{c} y da .$$

Substituting,

$$\tau = \frac{Q}{I_x b}\int_{y_0}^{c} y da . \qquad (12.13)$$

The integral in this last equation represents the first moment of the shaded cross - sectional area about the neutral axis of the beam. This area is always the portion of the cross section that is above the level at which the desired shear acts. This first moment of area is sometimes denoted by S_x in which case the above formula becomes:

$$\tau = \frac{QS_x}{bI_x}. \tag{12.14}$$

The units of $\int_{y_0}^{c} yda$ or of S_x are in^3 or m^3.

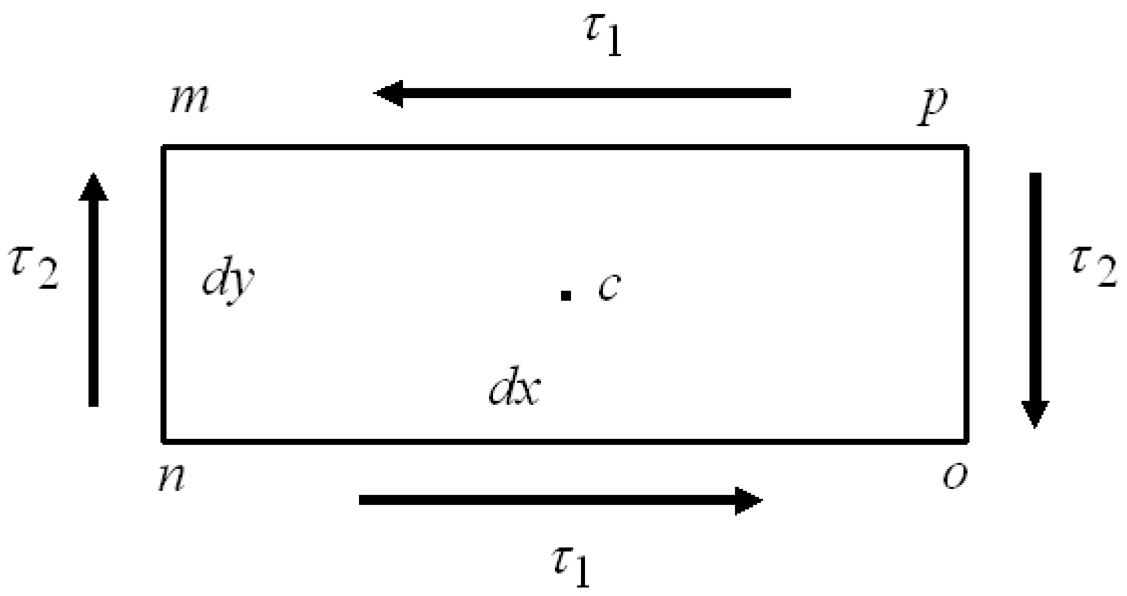


Fig. 12.8

The shearing stress τ just determined acts horizontally as shown in Fig. 12.7. However, let us consider the equilibrium of a thin element *nmop* of thickness t cut from any body and subject to a shearing stress τ_1 on its lower face, as shown in Fig. 12.8. The total horizontal force on the lower face is $\tau_1 tdx$. For equilibrium of forces in the horizontal direction, an equal force but acting in the opposite direction must act on the upper face, hence the shear stress intensity there too is τ_1. These two forces give rise to a couple of magnitude $\tau_1 tdxdy$. The only way in which equilibrium of the element can be maintained is for another couple to act over the vertical faces. Let the shear stress intensity on these faces be denoted by τ_2. The total force on either vertical face is $\tau_2 tdy$. For equilibrium of the moments about the center of the element we have:

$$\sum M_{c_i} = \tau_1 tdxdy - \tau_2 tdxdy = 0,$$

or

$$\tau_1 = \tau_2.$$

Thus we have the interesting conclusion that the shearing stresses on any two perpendicular planes through a point on a body are equal. Consequently, not only are there shearing stresses τ acting horizontally at any point in the beam, but shearing stresses of an equal intensity also act vertically at that same point.

In summary, when a beam is loaded by transverse forces, both horizontal and vertical shearing stresses arise in the beam. The vertical shearing stresses are of such magnitudes that their resultant at any cross section is exactly equal to the shearing force Q at that same section.

The integral in (12.12) represents the first moment of the shaded area of the cross section about the neutral axis. More generally, the integral always represents the first moment about the neutral axis of that part of the cross-sectional area of the beam between the horizontal plane on which the shearing stress τ occurs and the outer fibers of the beam, i.e., the area between y_0 and c.

From (12.12) it is evident that the maximum shearing stress always occurs at the neutral axis of the beam, whereas the shearing stress at the outer fibers is always zero. In contrast, the normal stress varies from zero at the neutral axis to a maximum at the outer fibers.

On beginning