Lecture 18
CASTIGLIANO'S THEOREM

Plan

1. General theory.
2. The example of application.

18.1. General theory.

Strain energy methods are particularly well suited to problems involving several structural members at various angles to one another. The fact that the members may be curved in their planes presents no additional difficulties. One of the great advantages of strain energy methods is that independent coordinate systems may be established for each member without regard for consistency of positive directions of the various coordinate systems. This advantage is essentially due to the fact that the strain energy is always a positive scalar quantity, and hence algebraic signs of external forces need be consistent only within each structural member.

Let us determine the internal strain energy stored within an elastic bar subject to an axial tensile force F.

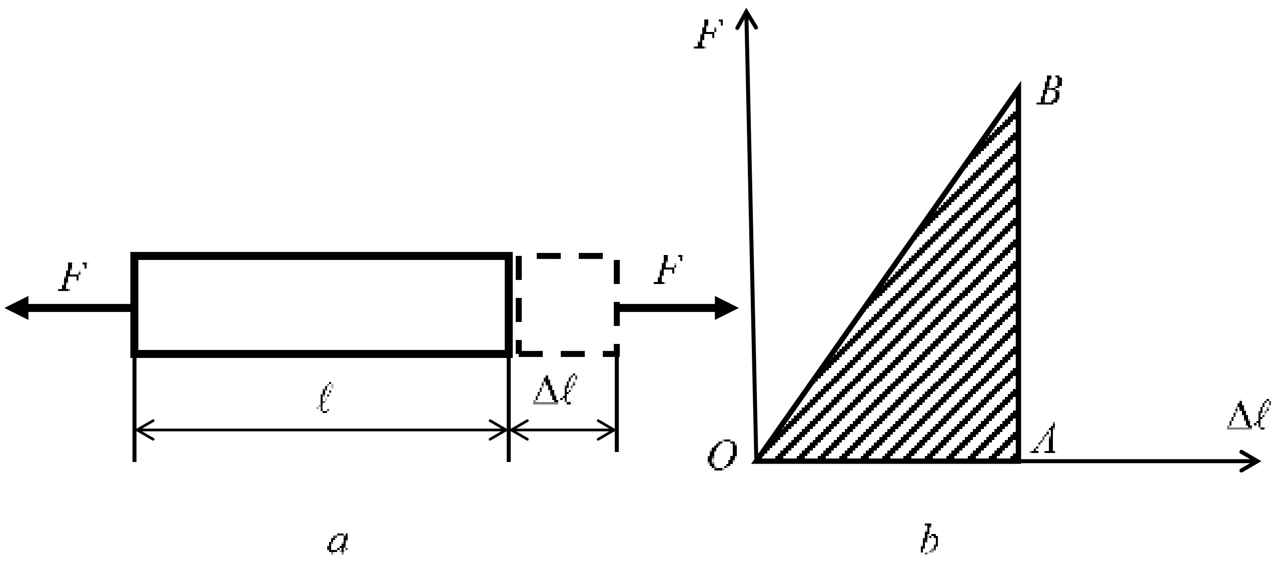

Fig. 18.1

For such a bar the elongation $\Delta\ell$ has been found that:

$$\Delta\ell = \frac{P\ell}{EA},$$

where A represents the cross - sectional area, ℓ is the length, and E is Young's modulus. The force - elongation diagram will consequently be linear, as shown in Fig. 18.1. For any specific value of the force P, such as that corresponding to point B in the force - elongation diagram, the force will have done positive work indicated by the shaded area OBA. This triangular area is given by $\frac{1}{2}P\Delta\ell$. Replacing $\Delta\ell$ by the value given above, this becomes $\frac{P^2\ell}{2EA}$. This is the work done by the external force and the work is stored within the bar in the form of internal strain energy, denoted by U. Hence:

$$U = \frac{P^2\ell}{2EA}.$$

Essentially, the elastic bar is acting as a spring to store this energy. The same expression for internal strain energy applies if the load is compressive, since the axial force appears as a squared quantity and hence the final result is the same for either a positive or negative force.

If the axial force P varies along the length of the bar, then in an elemental length dx of the bar the strain energy is:

$$dU = \frac{P^2 dx}{2EA},$$

and the energy in the entire bar is found by integrating over the length:

$$U = \int_0^\ell \frac{P^2 dx}{2EA}. \qquad (18.1)$$

Thus, when an external force acts upon an elastic body and deforms it, the work done by the force is stored within the body in the form of strain energy. The strain energy is always a scalar quantity. For a straight bar subject to a tensile force P, the internal strain energy U is given by:

$$U = \frac{P^2\ell}{2EA}, \qquad (18.2)$$

where ℓ represents the length of the bar, A is its cross - sectional area, and

E is Young's modulus.

Let us determine the internal strain energy stored within an elastic bar subject to a torque T as shown in Fig. 18.2.

The angle of twist θ has been found to be:

$$\theta = \frac{T\ell}{GI_\rho},$$

where G is the modulus of elasticity in shear, ℓ is the length, and I_ρ is the polar moment of inertia of the cross - sectional area. According to this expression, the relation between torque and angle of twist is a linear one, as shown in Fig. 18.2. When the torque has reached a specific value such as that indicated by point B, it will have done positive work indicated by the shaded area OBA. This triangular area is given by $\frac{1}{2}T\theta$,or $\frac{T^2\ell}{2GI_\rho}$. This work done by the external torque is stored within the bar as internal strain energy, denoted by U. Hence

$$U = \frac{T^2\ell}{2GI_\rho}.$$

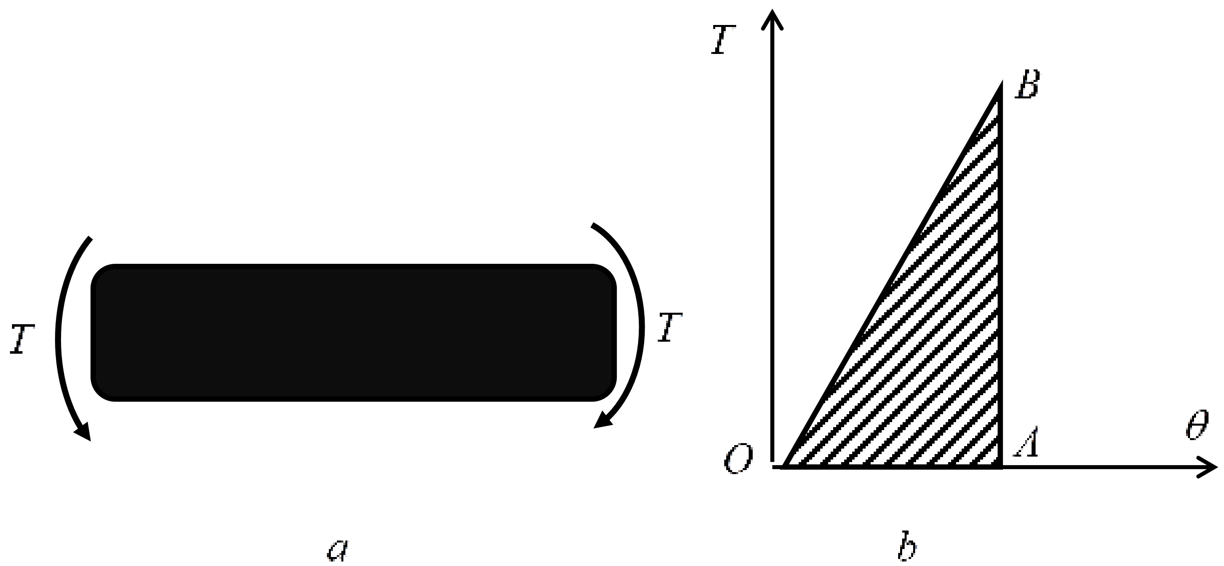

Fig. 18.2

If the torque T varies along the length of the bar, then in an elemental length dx the strain energy is:

$$dU = \frac{T^2 dx}{2GI_\rho},$$

and in the entire bar it is:

$$U = \int_0^\ell \frac{T^2 dx}{2GI_\rho}. \qquad (18.3)$$

Thus, for a circular bar of length ℓ subject to a torque T, the internal strain energy U is given by:

$$U = \frac{T^2 \ell}{2GI_\rho}, \qquad (18.4)$$

where G is the modulus of elasticity in shear and I_ρ is the polar moment of inertia of the cross-sectional area.

Let us determine the internal strain energy stored within an elastic bar subject to a pure bending moment M.

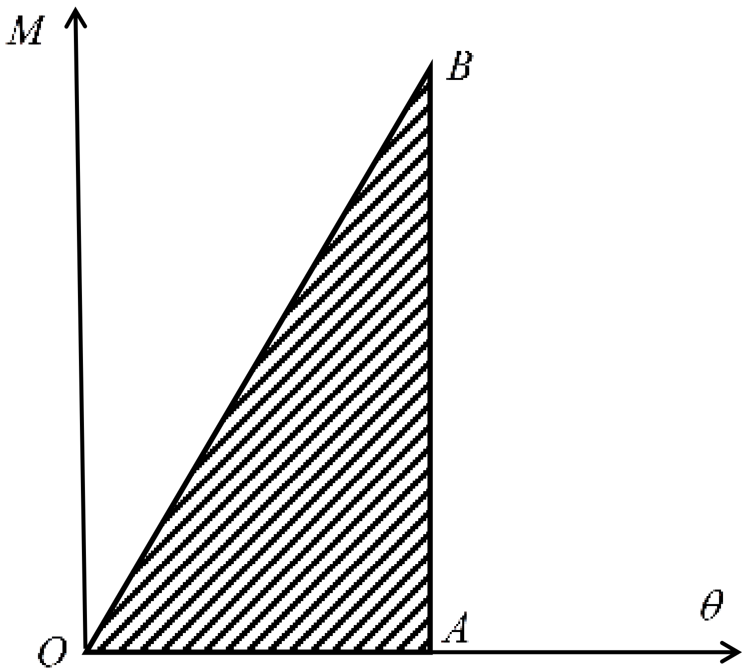

Fig. 18.3

An initially straight bar subject to the pure bending moment M which deforms it into a circular arc of radius of curvature ρ. It was shown that:

$$M = \frac{EI_x}{\rho},$$

where I_x denotes the moment of iner1ia of the cross - sectional area about the neutral axis. But the length of the bar, ℓ is equal to the product of the central angle θ subtended by the circular are and the radius ρ. Thus:

$$\frac{M}{EI_x} = \frac{1}{\rho} = \frac{\theta}{\ell},$$

or

$$\theta = \frac{M\ell}{EI_x}.$$

According to this the relation between moment and angle subtended is a linear one, and this is illustrated in Fig. 18.3. When the moment has reached a specific value M, such as that indicated by point B, it will have done work indicated by the shaded area OAB. This area is given by $\frac{1}{2}M\theta$ or $\frac{M^2\ell}{2EI_x}$. This work done by the external moment is stored within the bar as internal strain energy, denoted by U. Hence:

$$U = \frac{M^2\ell}{2EI_x}.$$

If the bending moment M varies along the length of the bar, then in an elemental length dx the strain energy is:

$$dU = \frac{M^2 dx}{2EI_x}, \qquad (18.5)$$

and in the entire bar it is:

$$U = \int_0^\ell \frac{M^2 dx}{2EI_x}. \qquad (18.6)$$

Thus, for a bar of length ℓ subject to a bending moment M, the internal strain energy U is given by

$$U = \frac{M^2\ell}{2EI_x}, \qquad (18.7)$$

where I_x is the moment of inertia of the cross - sectional area about the neutral axis.

Note that in each of these expressions the external load always occurs in the form of a squared magnitude, hence each of these energy expressions is always a positive scalar quantity.

Let us consider a general three - dimensional elastic body loaded by the forces P_1, P_2, etc. (Fig. 18.4). These would include forces exerted on the body by the various supports. We shall denote the displacement under P_1 in the direction of P_1 by δ_1, that under P_2 in the direction of P_2 by δ_2, etc. If we assume that all forces are applied simultaneously and gradually increased from zero to their final values given by P_1, P_2, etc., then the work done by the totality of forces will be:

$$U = \frac{P_1}{2}\delta_1 + \frac{P_2}{2}\delta_2 + \frac{P_3}{2}\delta_3 + \cdots. \qquad (18.8)$$

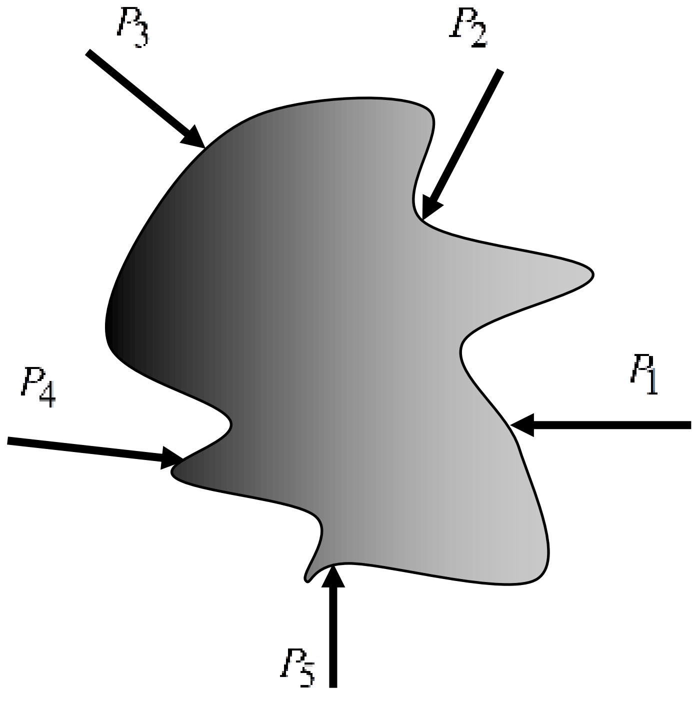

Fig. 18.4

This work is stored within the body as elastic strain energy.

Let us now increase the n-th force by an amount dP_n. This changes both the state of deformation and also the internal strain energy slightly. The increase in the latter is given by:

$$\frac{\partial U}{\partial P_n}\partial P_n. \qquad (18.9)$$

Thus, the total strain energy after the increase in the n-th force is:

$$U + \frac{\partial U}{\partial P_n} \partial P_n. \tag{18.10}$$

Let us reconsider this problem by first applying a very small force dP_n alone to the elastic body. Then, we apply the same forces as before, namely, P_1, P_2, P_3, etc. Due to the application of dP_n there is a displacement in the direction of dP_n which is infinitesimal and may be denoted by $d\delta_n$. Now, when P_1, P_2, P_3, etc., are applied, their effect on the body will not be changed by the presence of dP_n and the internal strain energy arising from application of P_1, P_2, P_3, etc., will be that indicated in (18.8). But as these forces are being applied the small force dP_n goes through the additional displacement δ_n caused by the forces P_1, P_2, P_3, etc. Thus, it gives rise to additional work $dP_n \cdot \delta_n$, which is stored as internal strain energy and hence the total strain energy in this case is:

$$U + dP_n \cdot \delta_n. \tag{18.11}$$

Since the final strain energy must be independent of the order in which the forces are applied, we may equate (18.10) and (18.11):

$$U + \frac{\partial U}{\partial P_n} \partial P_n = U + dP_n \cdot \delta_n,$$

or

$$\delta_n = \frac{\partial U}{\partial P_n}. \tag{18.12}$$

This is Castigliano's theorem: ***the displacement of an elastic body under the point of application of any force, in the direction of that force, is given by the partial derivative of the total internal strain energy with respect to that force***. Equations for U are given in (18.1), (18.3) and (18.6) for axial, torsional, and bending loadings, respectively. However, instead of using the integral forms of the equations in those problems, it is usually more convenient to differentiate through the integral signs, and thus for a body subject to combined axial, torsional and bending effects, we have for the displacement δ_n under the force P_n:

$$\delta_n = \int_\ell \frac{P}{AE}\frac{\partial P}{\partial P_n}dx + \int_\ell \frac{T}{GI_\rho}\frac{\partial T}{\partial P_n}dx + \int_\ell \frac{M}{GI_x}\frac{\partial M}{\partial P_n}dx. \quad (18.13)$$

For a body composed of a finite number of elastic subbodies, these in1egrals are replaced by finite summations.

The term "force" here is used in its most general sense and implies either a true force or a couple. For the case of a couple, Castigliano's theorem gives the angular rotation under the point of application of the couple in the sense of rotation of the couple.

It is importanl to observe that the above derivation required that we be able to vary the n-th force, P_n, independently of the other forces. Thus, P_n must be statically independent of the other external forces, implying that the energy U must always be expressed in terms of the statically independent forces of the system. Obviously, reactions that can be determined by statics cannot be considered as independent forces.

Thus, this theorem is extremely useful for finding displacements of elastic bodies subject to axial loads, torsion, bending, or any combination of these loadings. The theorem states that the partial derivative of the total internal strain energy with respect to any external applied force yields the displacement under the point of application of that force in the direction of that force. Here, the terms force and displacement are used in their generalized sense and could either indicate a usual force and its linear displacement, or a couple and the corresponding angular displacement. In equation form the displacement under the point of application of the force P_n is given according to this theorem by (18.12).

In such problems all external reactions can be found by application of the equations of statics. After this has been done, the deflection under the point of application of any external applied force can be found directly by use of Castigliano's theorem. If the deflection is desired at some point where there is no applied force, then it is necessary to introduce an auxiliary (i.e., fictitious) force at that point and. treating that force just as one of the real ones, use Castigliano's theorem to determine the deflection at that point. At the end of the problem the auxiliary force is set equal to zero.

Castigliano's theorem is extremely useful for determining the indeterminate reactions in such problems. This is because the theorem can be applied to each reaction, and the displacement corresponding to each reaction is known beforehand and is usually zero. In this manner it is possible to establish as many equations as there are redundant reactions, and these equations together with those found from statics yield the solution for all reactions. After the values of all reactions have been found, the deflection at any desired point can be found by direct use of Castigliano's theorem.

On beginning

18.2. The example of application.

Determine by Castigliano's integral the deflection at point B of the cantilever beam subject to the single concentrated force P, as shown in Fig. 18.5.

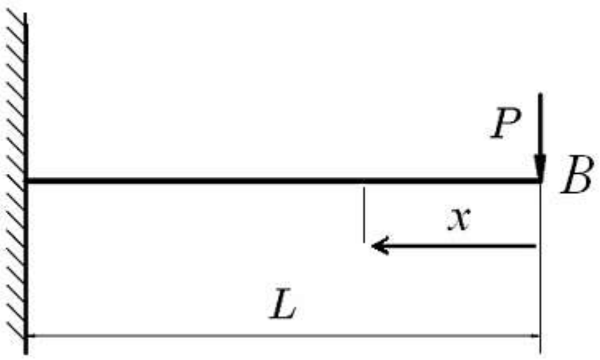

Fig. 18.5

Hence the bending moment M at the section x is:

$$M_x = -P \cdot x .$$

Than the partial derivative of bending moment by force P is:

$$\frac{\partial M_x}{\partial P} = -x .$$

Let us written the Castigliano's integral:

$$\delta_B = \int_0^L \frac{-P \cdot x}{EI_x}(-x)dx = \int_0^L \frac{P \cdot x^2}{EI_x}dx = \frac{P \cdot x^3}{3EI_x}\Bigg|_0^L = \frac{PL^3}{3EI_x} .$$

We obtained the result of problem, which is the analogical decision of problem in lectures 16 and 17. It is differs only a sign, because the x -axis has opposite direction.

On beginning