Lecture 26
COMPLEX BENDING

Plan

1. General theory.
2. Example of calculation of beam on complex bending.

26.1 General theory

We considered the plane bending of beams, in which all loads lay in one plane passing through one of the main axes of the cross-section. For such a bending, the axis deforms in the plane of loading.

The bending of beam, in which the line of action of load does not coincident with one of the main axes of the cross-section, is called **complex bending**.

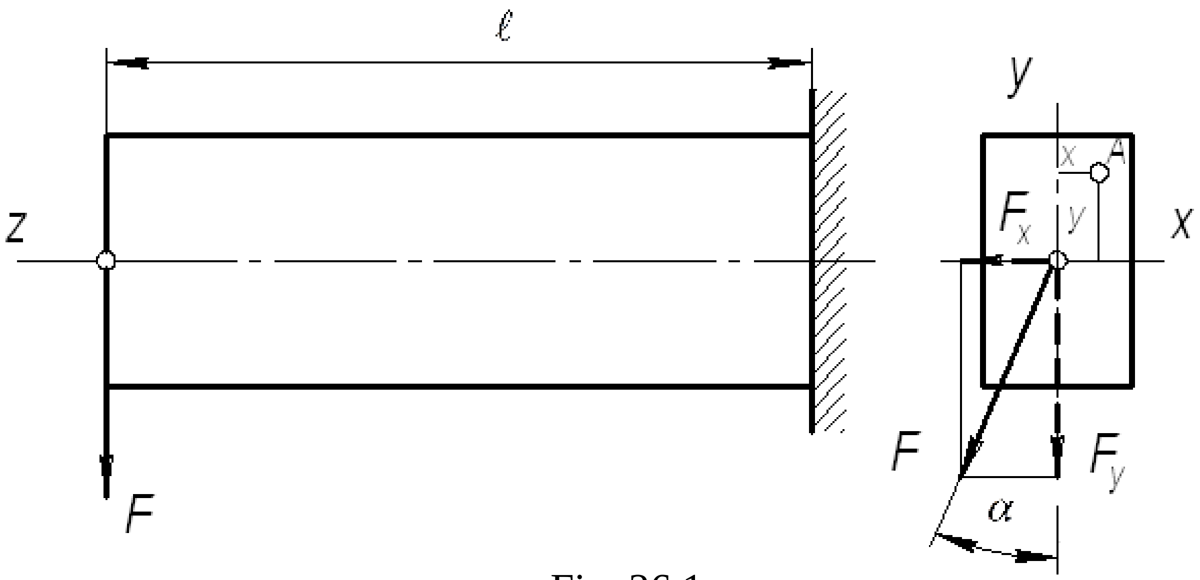

Fig. 26.1

Consider the cantilever beam with length ℓ, which has a rectangular cross-section. The free end of beam is loaded by force F, which forms angle α with an axis y (fig. 26.1)

Let us expand the force into two components, which will be directed along the main axes of the cross-section. Using the principle of the independence of the forces action, we will reduce the case of complex bending to case of direct bending, which comes about in two mutually perpendicular planes. It is obvious, that the dangerous cross-section will be in point in which the beam is fixed. Then the maximum bending moments will be equal:

$$M_x = F\ell \cos\alpha; \qquad M_y = F\ell \sin\alpha.$$

Corresponding to these bending moments, the normal stress at point A of a dangerous cross-section is calculated by the formulas:

$$\sigma_{1A} = \frac{M_{3x} \cdot y}{I_x}. \qquad \sigma_{2A} = \frac{M_{3y} \cdot x}{I_y},$$

where x and y are current coordinates of point A; I_x and I_y are moments of inertia of the cross section relative to neutral axis.

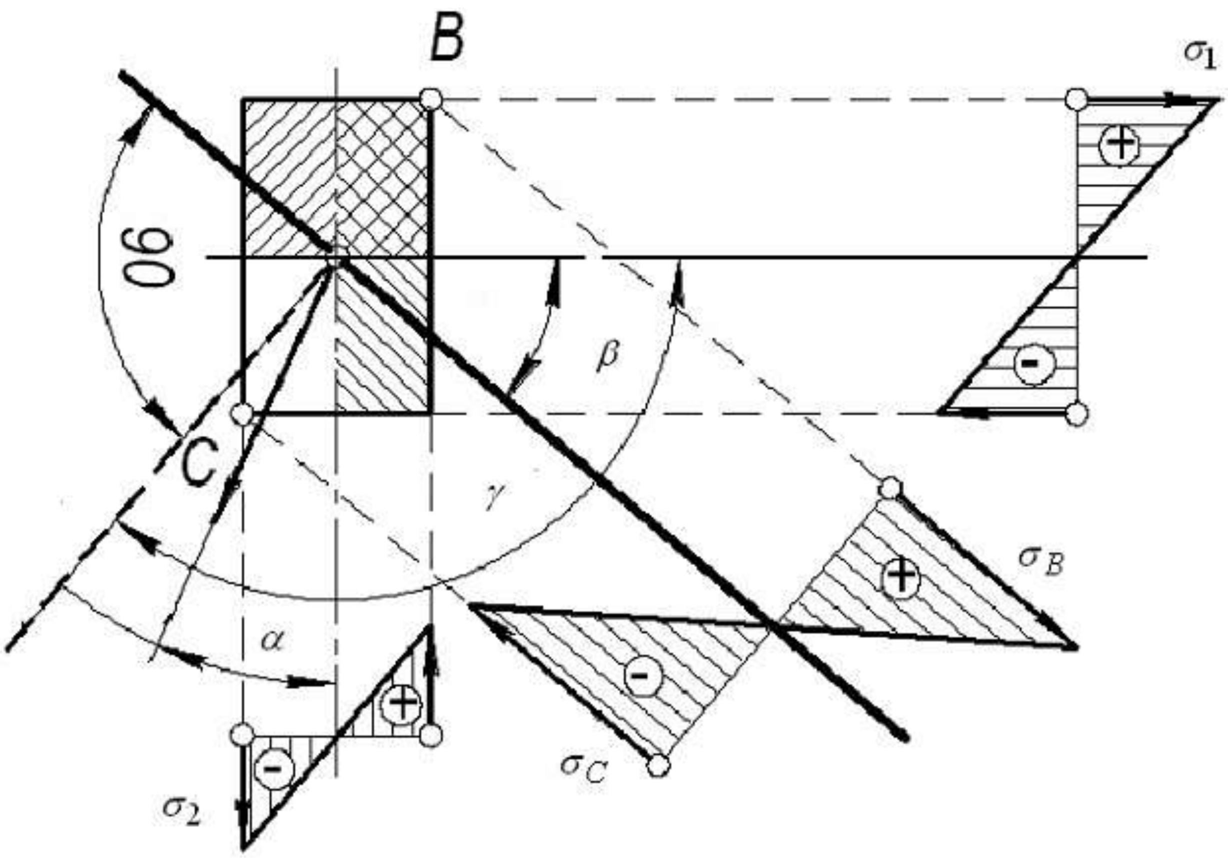

Fig.

The total normal stresses at the point A will be equal:

$$\sigma_A = \sigma_{1A} + \sigma_{1A} = \frac{M_x \cdot y}{I_x} + \frac{M_y \cdot x}{I_y} = F\ell\left(\frac{y\cos\alpha}{I_x} + \frac{x\sin\alpha}{I_y}\right). \qquad (26.1)$$

It is obvious that the maximum tensile stresses occurs at the point B, and the maximum compression stresses occurs at the point C of the dangerous section (fig. 26.2). The diagrams of normal stresses are shown on the same figure.

Since the condition on the neutral axis is that $\sigma_A = 0$, then its equation has the form:

$$\frac{y\cos\alpha}{I_x} + \frac{x\sin\alpha}{I_y} = 0,$$

where x and y are current coordinates of the neutral axis.

From the equation one can see that the neutral axis is a straight line which passes through the origin of the coordinate, that is, through the centroid of the beam. Determine the angle β, that the neutral axis forms with the axis x:

$$tg\beta = \frac{y}{x} = -\frac{I_x \sin\alpha}{I_y \cos\alpha} = -\left(\frac{I_x}{I_y}\right) tg\alpha. \qquad (26.2)$$

From this equation shows that when

$$I_x \neq I_y,$$

then $\beta \neq \alpha$ and the neutral axis is not perpendicular to the line of force F.

Using the principle of the independence of the forces, determine the direction of bending the beam under the acting of force F. The deflection f_x in the direction of the axis x will be equal:

$$f_x = \frac{F\ell^3 \sin\alpha}{3EI_y}.$$

The deflection f_y in the direction of the axis y will be equal:

$$f_y = \frac{F\ell^3 \cos\alpha}{3EI_x}.$$

The total deflection is determined by the formula:

$$f = \sqrt{f_x^2 + f_y^2}.$$

Denote the angle γ between the direction of the total deflection and the axis x. Then we will have:

$$ctg\gamma = \frac{f_x}{f_y} = \frac{\dfrac{F\ell^3 \sin\alpha}{3EI_y}}{\dfrac{F\ell^3 \cos\alpha}{3EI_x}},$$

or

$$ctg\gamma = \left(\frac{I_x}{I_y}\right) tg\alpha . \tag{26.3}$$

Comparing expressions (7.7) with (7.8), we arrive at the conclusion that they differ only in signs, therefore, the angles themselves differ by 90°. This means that the total deflection of beam lies in a plane perpendicular to the neutral axis. From this we can conclude that **the deflection plane of the complex bending does not coincide with the plane of loading.**

On beginning

26.2. Example of calculation of beam on complex bending.

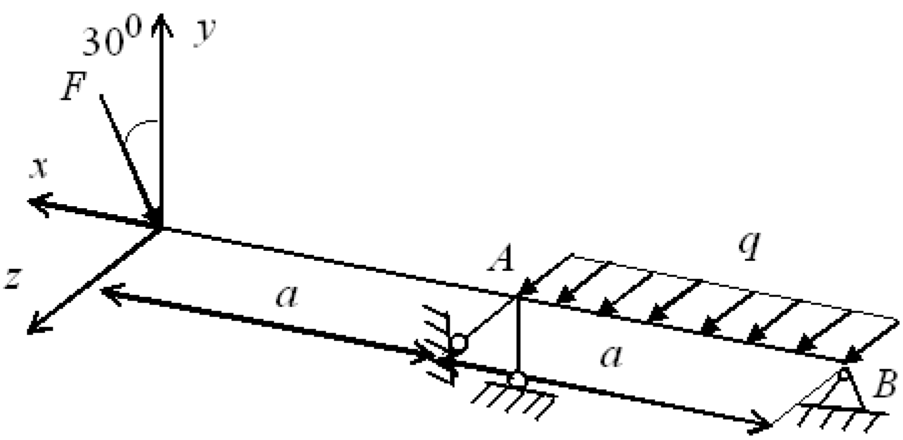

Fig. 26. 3

The force of $F = 14$ kN, which with the vertical axis y forms an angle, which is equal 30^0 (Fig. 26. 3), is applied to the free end of the hinge supported beam. The uniformly distributed loads, that equal $q = 12$ kN/m is acting on the portion of beams, which has length of $AB = a = 2$ m in the plane xz.

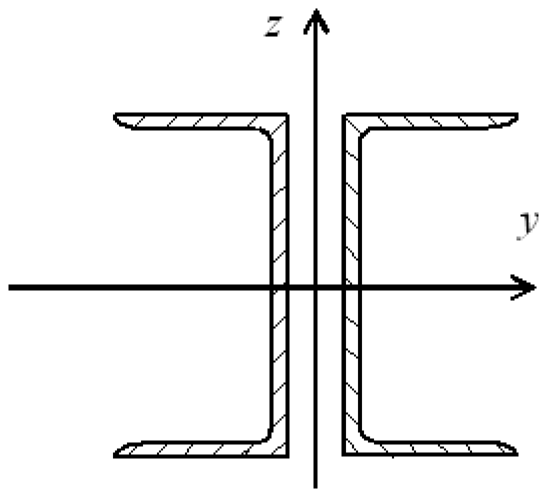

Fig. 26.4

Build a bending moment diagrams in the horizontal and vertical planes. Choose the rational location of a given cross-section (Fig. 26.4).

Choose the number of double T-beam from the condition of strength under normal stresses by the method of successive approximations, if the material from which the beam is made is steel 20. The coefficient of yield strength is equals 1,5.

Let us find projections of force F in the direction of the axes y and z. We have to write:

$$F_y = F \cdot \cos 30^0 = 14 \cdot 0{,}866 = 12{,}124 \text{ kN};$$

$$F_z = F \cdot \sin 30^0 = 14 \cdot 0{,}5 = 7 \text{ kN}.$$

An external load is applied to the beam in each of the planes, which are xy and xz. Find the reactions of beam support and build appropriate bending moment diagrams.

1. Let us consider the plane xz.

We will determine the support reaction of beam, that is shown on fig. 26.5, a.

1. $\sum M_{Ay} = 0$. Then $F_y \cdot a + R_{By} \cdot a = 0$,

where we have: $R_{By} = -F_y = -12{,}124$ kN;

2. $\sum M_{By} = 0$. Then: $F_y \cdot 2a - R_{Ay} \cdot a = 0$,

where we have: $R_{Ay} = 2F_y = 24{,}248$ kN.

Verification of the reactions of supports of beams, that was determinated:

$$\sum F_{yi} = 0; \quad -F_y + R_{Ay} + R_{By} = -12{,}124 + 24{,}248 - 12{,}124 =$$
$$= 24{,}248 - 24{,}248 = 0.$$

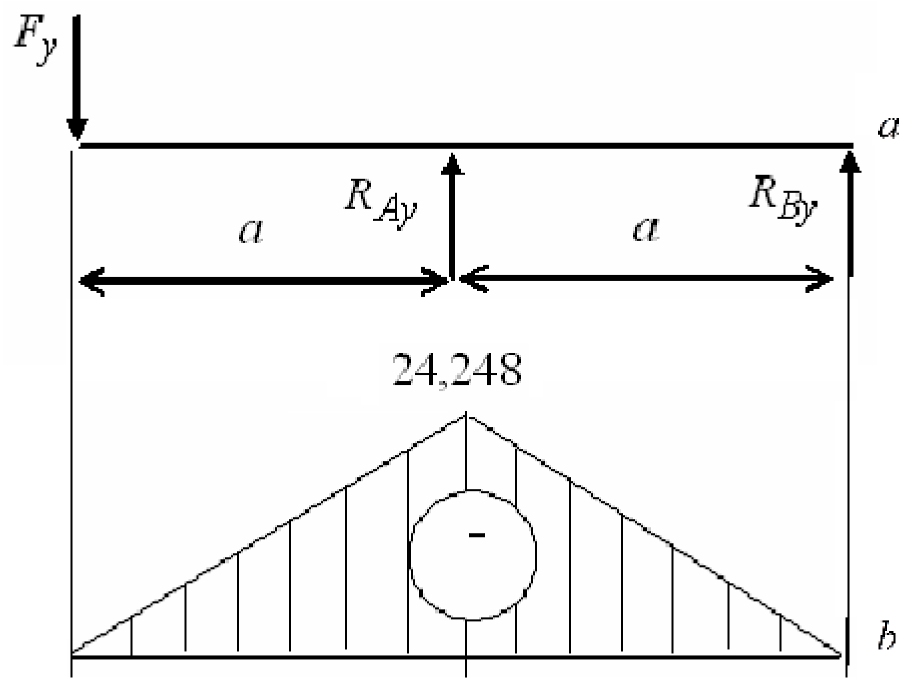

Fig. 26.5

Draw the diagram of the bending moment M_z, which is presented on fig. 26.5, b:

I portion $0 \le x_1 \le a$. $\quad M_{z1} = -F_y \cdot x_1;$

$$M_{z1}(0) = 0; \qquad M_{z1}(a) = -12{,}124 \cdot 2 = -24{,}248 \text{ kNm.}$$

II portion $0 \le x_2 \le a$. $\quad M_{z2} = R_{By} \cdot x_2;$

$$M_{z2}(0) = 0; \qquad M_{z2}(a) = -12{,}124 \cdot 2 = -24{,}248 \text{ kNm.}$$

2. Let us consider the plane yz.

We will determine the support reaction of beam, that is shown on fig. 26.6, a.

1. $\sum M_{Az} = 0$. Then: $F_z \cdot a + R_{Bz} \cdot a + \frac{qa^2}{2} = 0$,

where we have: $R_{Bz} = F_z - \frac{qa}{2} = 7 - \frac{12 \cdot 2}{2} = -5$ kN;

2. $\sum M_{Bz} = 0$. Then: $-F_z \cdot 2a - R_{Az} \cdot a - \frac{qa^2}{2} = 0$,

where we have: $R_{Az} = -2F_z - \frac{qa}{2} = -2 \cdot 7 - \frac{12 \cdot 2}{2} = -26$ kN.

Verification of the reactions of supports of beams, that was determinated:

$$\sum F_{zi} = 0; \quad F_z + R_{Az} + R_{Bz} + q \cdot a = 7 - 26 - 5 + 12 \cdot 2 =$$
$$= 31 - 31 = 0.$$

Draw the diagram of the bending moment M_y, which is presented on fig. 26.6, b:

I portion $0 \le x_1 \le a$. $M_{y_1} = F_z \cdot x_1$;

$$M_{y_1}(0) = 0; \qquad M_{y_1}(a) = 7 \cdot 2 = 14 \text{ kNm.}$$

II portion $0 \le x_2 \le a$. $M_{y_2} = R_{Bz} \cdot x_2 + \frac{qx_2^2}{2}$;

$$M_{y_2}(0) = 0; \qquad M_{y_2}(a) = -5 \cdot 2 + \frac{12 \cdot 2^2}{2} = 14 \text{ kNm.}$$

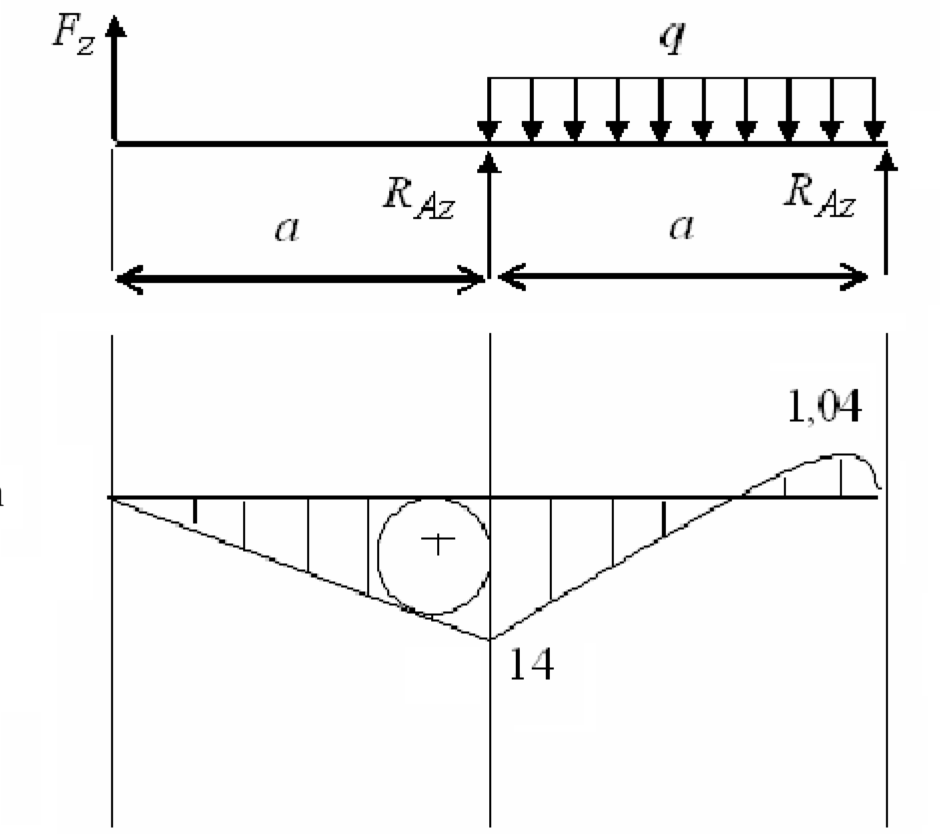

Diagram of M_y, kNm

Fig. 26.6

The dangerous cross-section of beam will be the cross-section A, where are $\left|M_{z_A}\right| = 24{,}248$ kNm and $\left|M_{y_A}\right| = 14$ kNm.

Then the allowable stresses for steel 20 are equal:

$$[\sigma] = \frac{\sigma_m}{n_m} = \frac{250}{1{,}5} = 166 \text{ MPa}.$$

From the condition of strength under normal stresses, we obtain:

$$W_z \geq \frac{M_z}{[\sigma]} = \frac{24{,}240 \cdot 10^3}{166 \cdot 10^6} = 146 \cdot 10^{-6} \text{ m}^3$$

or

$$W_z \geq 146 \text{ sm}^3.$$

Where

$$W_{z_i} = 0{,}5 \cdot W_z = 0{,}5 \cdot 146 = 73 \text{ cm}^3.$$

We select a channel number 16 from the table of assortment of channels, which has following characteristics: $W_z = 93{,}4$ sm^3; $A = 18{,}1$ cm^2, $I_y = 63{,}3$ cm^4; $z_0 = 1{,}8$ cm; $b = 6{,}4$ cm.

Then we can write, that:

$$W_z = 2 \cdot W_{z_i} = 2 \cdot 93{,}4 = 186{,}6 \text{ cm}^3.$$

$$I_y = 2\left(I_{y_i} + z_0^2 \cdot A_i\right) = 2\left(63{,}3 + 1{,}8^2 \cdot 18{,}1\right) = 243{,}9 \text{ cm}^4.$$

Where:

$$W_y = \frac{I_y}{z_{\max}} = \frac{243{,}9}{6{,}4} = 38{,}1 \text{ cm}^3.$$

Determine the maximum stresses that arise in the cross section:

$$\sigma_{\max} = \frac{\left|M_{z_A}\right|}{W_z} + \frac{\left|M_{y_A}\right|}{W_y} = \frac{14 \cdot 10^3}{38 \cdot 10^{-6}} + \frac{24{,}240 \cdot 10^3}{186{,}6 \cdot 10^{-6}} \approx 497 \cdot 10^6 \text{ Pa},$$

or

$$\sigma_{\max} = 497 \text{ MPa} > [\sigma] = 166 \text{ MPa}.$$

We select a channel number 24 from the table of assortment of channels, which has following characteristics: $W_z = 242$ cm^3; $A = 30{,}6$ cm^2, $I_y = 208$ cm^4; $z_0 = 2{,}42$ cm; $b = 9$ cm.

Then we can write, that:

$$W_z = 2 \cdot W_{z_i} = 2 \cdot 242 = 484 \text{ cm}^3.$$

$$I_y = 2\left(I_{y_i} + z_0^2 \cdot A_i\right) = 2\left(208 + 2{,}42^2 \cdot 30{,}6\right) = 774 \text{ cm}^4.$$

Where:

$$W_y = \frac{I_y}{z_{\max}} = \frac{774}{9} = 86 \text{ cm}^3.$$

Determine the maximum stresses that arise in the cross section:

$$\sigma_{\max} = \frac{\left|M_{z_A}\right|}{W_z} + \frac{\left|M_{y_A}\right|}{W_y} = \frac{14 \cdot 10^3}{86 \cdot 10^{-6}} + \frac{24{,}240 \cdot 10^3}{484 \cdot 10^{-6}} \approx 213 \cdot 10^6 \text{ Pa},$$

or

$$\sigma_{\max} = 213 \text{ MPa} > [\sigma] = 166 \text{ MPa}.$$

We select a channel number 30 from the table of assortment of channels, which has following characteristics: $W_z = 387$ cm^3; $A = 40{,}5$ cm^2, $I_y = 327$ cm^4; $z_0 = 2{,}52$ cm; $b = 10$ cm.

Then we can write, that:

$$W_z = 2 \cdot W_{z_i} = 2 \cdot 387 = 774 \text{ cm}^3.$$

$$I_y = 2\left(I_{y_i} + z_0^2 \cdot A_i\right) = 2\left(327 + 2{,}52^2 \cdot 40{,}5\right) = 1168 \text{ cm}^4.$$

Where:

$$W_y = \frac{I_y}{z_{\max}} = \frac{1168}{10} = 116{,}8 \text{ cm}^3.$$

Determine the maximum stresses that arise in the cross section:

$$\sigma_{\max} = \frac{\left|M_{z_A}\right|}{W_z} + \frac{\left|M_{y_A}\right|}{W_y} = \frac{14 \cdot 10^3}{116{,}8 \cdot 10^{-6}} + \frac{24{,}240 \cdot 10^3}{7746 \cdot 10^{-6}} \approx 151 \cdot 10^6 \text{ Pa},$$

or

$$\sigma_{\max} = 151 \text{ MPa} < [\sigma] = 166 \text{ MPa}.$$

The deficiency of loading of cross section is:

$$\varepsilon = \frac{166 - 151}{166} \cdot 100\% = 9 \text{ \%}.$$

On beginning