Lecture 27
THE OFF-CENTRE ACTING OF FORCE OF TENSION OR COMPRESSION

Plan

1. General theory.
2. The core of the cross-section.
3. Example of calculation of beam on the off-centre acting of force.

27.1 General theory

In practice, the case where the longitudinal loading is applied not in the centre of the weight of the cross-section of the beam, but at a certain distance from it is quite often encountered. This distance is called the **eccentricity relative to the principal axes of inertia**. Such problems are often encountered in bridge construction when calculating bridge supports, in civil engineering at calculations of columns of buildings, etc.

Assume that the compressive force F is applied at the point (Fig. 27.1) with coordinates relative to the central axis of inertia of the cross section z_F and y_F.

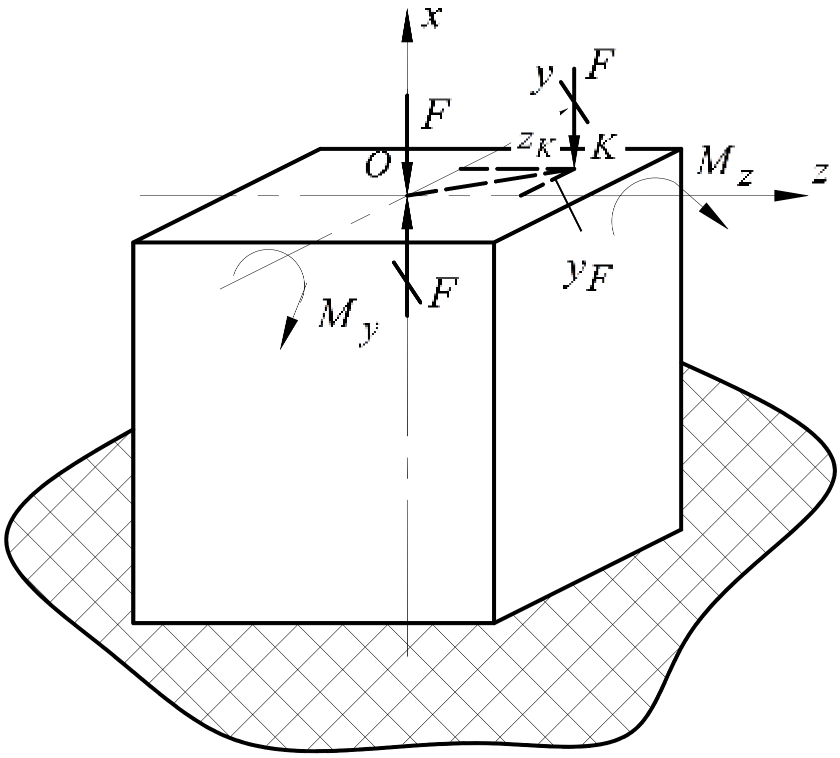

Fig 27.1

Let us give the force F to the centre of weight of cross-section. To do this, we load the beam by two forces which are equal in magnitude F and which have opposite directions, in point O. As a result, the central force F will cause compression stresses in arbitrary cross-section of the bar.

In this case, two forces, which are indicated by crossed lines in Fig. 27.1, form a pair with the moment:

$$M_O = F \cdot OK,$$

that the bar will bending.

If we decompose the moment M_O into two components relative to the axes z and y, we will obtain:

$$\mathrm{M}_z = F \cdot y_F, \quad \mathrm{M}_y = F \cdot z_F. \tag{27.1}$$

It is assumed that the bar is characterized by a sufficient rigidity. Accordingly, it is possible to neglect its deformations, which will lead to a change in the coordinates of force F.

We apply the principle of the independence of the forces acting on stresses at an arbitrary point of the cross-section with coordinates z and y. Then, in the case of the off-centre acting of longitudinal force, the stresses are determined from the dependence:

$$\sigma = -\frac{F}{A} - \frac{M_z \cdot y}{I_z} - \frac{M_y \cdot z}{I_y}, \tag{27.2}$$

or

$$\sigma = -\frac{F}{A} - \frac{F \cdot y_F \cdot y}{I_z} - \frac{F \cdot z_F \cdot z}{I_y}. \tag{27.3}$$

In the case of acting a tensile force the formula (27.3) will look like:

$$\sigma = \frac{F}{A} + \frac{F \cdot y_F \cdot y}{I_z} - \frac{F \cdot z_F \cdot z}{I_y}. \tag{27.4}$$

In general case, basing on the formulas (27.3) and (27.4), the formula for determining stresses in the case of off-centre action of longitudinal force can be written in the following form:

$$\sigma = \pm\frac{F}{A} \pm \frac{F \cdot y_F \cdot y}{I_z} \pm \frac{F \cdot z_F \cdot z}{I_y}. \qquad (27.5)$$

Given that $i_z = \sqrt{\frac{I_z}{A}}$ and $i_y = \sqrt{\frac{I_y}{A}}$, formula (27.5) can be represented as:

$$\sigma = \pm\frac{F}{A} \pm \frac{F \cdot y_F \cdot y}{i_z^2 \cdot A} \pm \frac{F \cdot z_F \cdot z}{i_y^2 \cdot A}.$$

If the multiplier $\frac{F}{A}$ to factor out, we will get:

$$\sigma = \frac{F}{A}\left(\pm 1 \pm \frac{y_F \cdot y}{i_z^2} \pm \frac{z_F \cdot z}{i_y^2} \right), \qquad (27.6)$$

where i_z^2 and i_y^2 are squares radii of inertia relative to the principal axes of inertia z and y.

Conditions strength of materials with different tensile and compression resistance is written as follows:

$$\sigma_{\max} = \frac{F}{A}\left(\pm 1 \pm \frac{y_F \cdot y}{i_z^2} \pm \frac{z_F \cdot z}{i_y^2} \right) \le \left[\sigma_p\right],$$

$$(27.7)$$

$$\sigma_{\max} = -\frac{F}{A}\left(\pm 1 \pm \frac{y_F \cdot y}{i_z^2} \pm \frac{z_F \cdot z}{i_y^2} \right) \le \left[\sigma_{cm}\right].$$

On beginning

27.2. The core of the cross-section.

As with complex bending, and with off-centre acting of longitudinal loading, it is very important to know the position of the neutral axis of the cross-section. Knowing the location of the neutral axis, it is easy to determine the dangerous points of the cross-section, based on the properties of the materials. For brittle materials (cast iron, concrete, etc.) is the most dangerous in terms of tensile stresses and for plastic is less dangerous within the allowable stresses.

In the equation (27.7) we denote the coordinates of the neutral axis of the cross section by z_0 and y_0. Equating the first part of the equation (27.7) to zero, we have:

$$1+\frac{y_F \cdot y_0}{i_z^2}+\frac{z_F \cdot z_0}{i_y^2}=0. \qquad (27.8)$$

From Equation (27.8) it follows that the neutral axis never passes through the centre of weight of cross-section. It cuts off the axes z and y the segments, which equal in magnitude a_z and a_y (see Fig. 27.2).

let us suppose, that $y_0=0$, the point of application of force F is on the axle z, then $z_0=a_z$.

From equation (27.8) we get:

$$a_z=-\frac{i_y^2}{z_F}. \qquad (27.9)$$

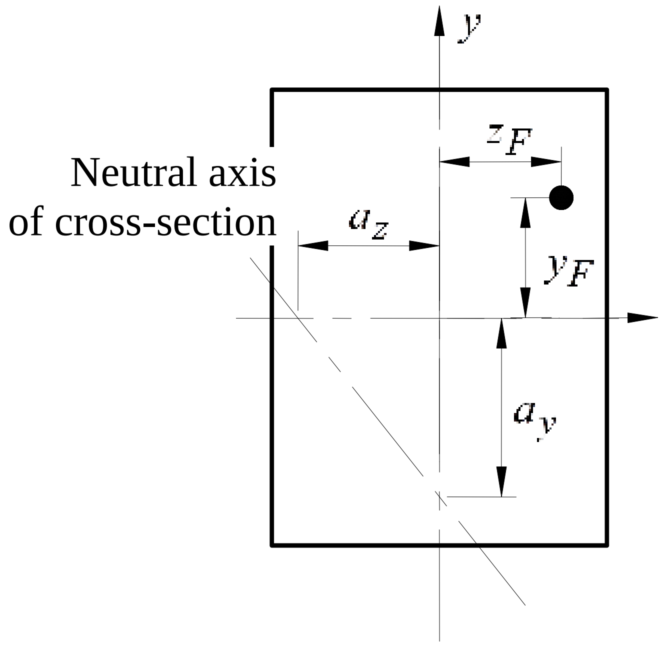

Fig 27.2

For the case when $z_0=0$, it is the case when the point of application of force F lies on the axis with similarly we have:

$$a_y = -\frac{i_z^2}{y_F}. \qquad (27.10)$$

From the formula (27.8) it follows that the position of the neutral axis of the cross section depends on its shape and size, and also on the coordinates of application of force F. At the same time, it absolutely does not depend on the magnitude of force F.

Some building materials (concrete, brickwork) and as well as machine-building (cast iron) can withstand only very small stresses of tension. Therefore, it is not desirable to use them for fabrication of structural elements working on bending, torsion, central and off-central tension.

It was found that if compressive force F, which is located not central relative to principal central axes of inertia, the neutral axis passes near the centroid of cross-section. Then in the cross-section will be stresses of tension and compression. But you can apply the compressive force in such a way that at arbitrary point of the cross-section only the stresses of compression will arise. This is possible only when the point of application of compressive stresses is in the middle of central part of cross- section, which is called the **core of the cross-section**.

Consequently, **the central part of the cross-section, in which or at its limit, the application of compressive force, causes only compressive stresses at all points of the cross-section is called the core of the cross-section.**

If the compressive force is applied outside the core of the cross section, then in the cross section there will be both tension and compressive stresses. Therefore, it is important to know the shape and size of the core of the cross-section when we calculate the off-centre compression of the beam which is made from materials that are poorly perceive stresses of tension.

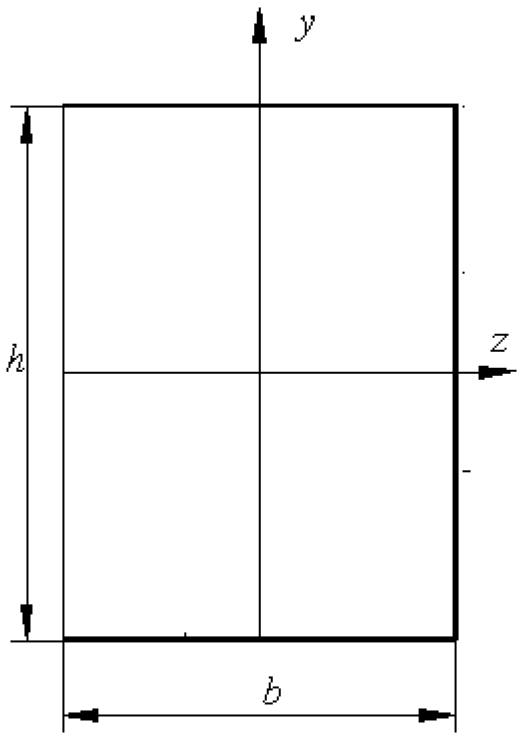

Fig. 27.3

In order to build the core of the cross-section, it is necessary to consider all possible variants of applying the compressive force. In this case, it is necessary to ensure that the neutral axis of the cross-section is tangent to the contour of the cross-section. And in no case it can not passes through the cross-section.

Let us consider the process of building the core of the cross-section on example of the rectangular cross-section, which is presented on Fig. 27.3.

Assume that the neutral axis of the cross-section at a certain point of force application is located on the contour *I* – *I* of the cross-section (Fig. 27.3). Then the first coordinate of the core of the cross- section is determined from formula (27.10):

$$y_F = -\frac{i_z^2}{a_y}.$$

Taking into account that $a_y = \frac{h}{2}$ and $i_z^2 = \frac{h^2}{12}$, we obtain:

$$y_F = -\frac{h^2}{12}\bigg/\frac{h}{2} = -\frac{h}{6}.$$

Thus, point 1 lies on the axis y at a distance which is $-\frac{h}{6}$. This distance is postponed in the negative direction of the coordinate axis y, as it is presented on Fig. 27.4.

Similarly, the point 3 for the tangent *III - III* will lie on the axis y and at a distance is $\frac{h}{6}$ also.

Similarly, we can find the vertices of the core of the cross-section relative to the tangents *II* – *II* and *IV* – *IV*. They will be equal accordingly $z_F = \pm\frac{b}{6}$.

To complete the construction of the core of the cross-section, you need to go around the whole contour of cross-section by tangent. This will be done by rotating tangents around the angular points of the section A_i (Fig. 27.4), which we have as poles. Each such rotation corresponds to a straight line, which is on the core of a cross-section. Thus, the cross-section will have the form of a rhomb, as shown in Fig. 27.4.

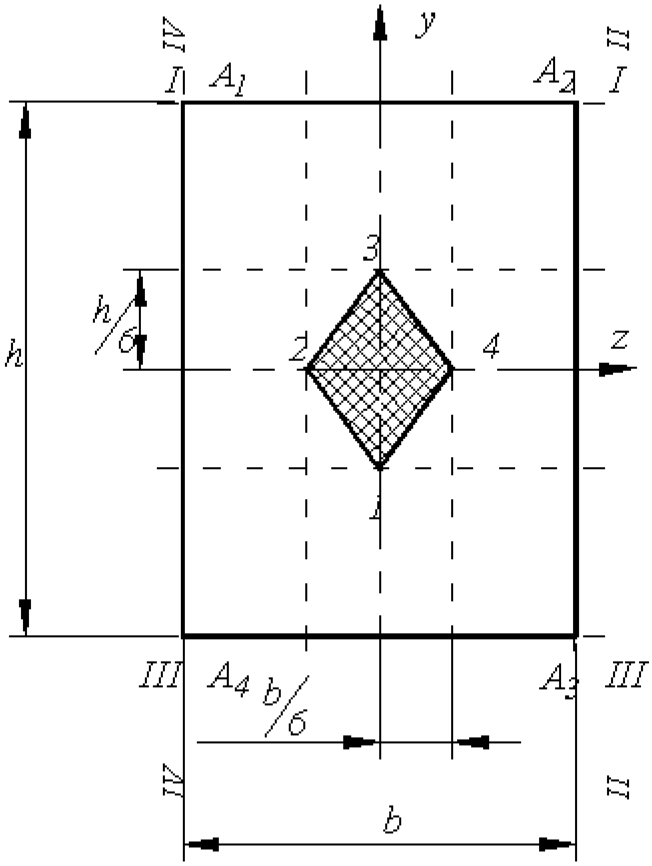

Fig. 27.4

The force of $F = 14$ kN, which with the vertical axis y forms an angle, which is equal 30^0 (Fig. 27. 3), is applied to the free end of the hinge supported beam. The uniformly distributed loads, that equal $q = 12$ kN/m is acting on the portion of beams, which has length of $AB = a = 2$ m in the plane xz.

On beginning

27.3. Example of calculation of beam on the off-centre acting of force.

For a given cross section, shown in Fig. 27.5, build the core of the cross-section, if $a = 3$ dm, $b = 2$ dm.

Check whether the force will be applied to the core of cross-section if its point of application is at the intersection of lines, which are shown dotted lines on Fig. 27.5.

First we will determine the coordinates of the point, where the force is applied. To do this, we construct the equations for the straight lines I and II, which are shown in Fig. 27.6.

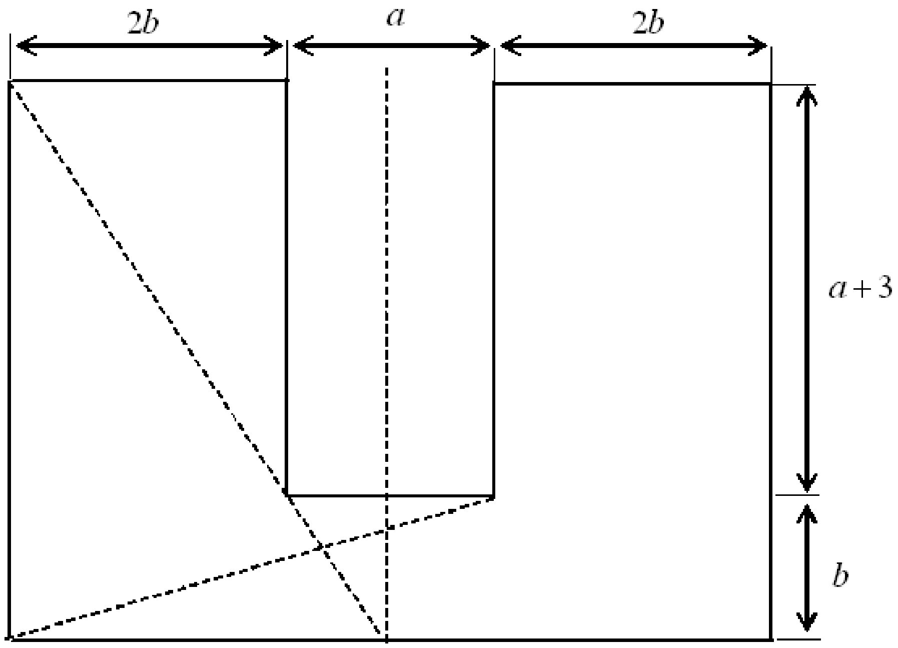

Fig. 27.6

To do this, we introduced a coordinate system O_1xy (Fig. 27.7). Then the equation of the line I will have the form:

$$y_1 = k_1 x.$$

From Fig. 27.7 it is seen that this line passes through a point with coordinates $(2b + a, b)$, we can write:

$$b = k_1(2b + a),$$

Where

$$k_1 = \frac{b}{2b + a}.$$

Consequently, the equation of the line I will be finally written as follows:

$$y_1 = \frac{b}{2b + a} x.$$

Equation of the line II will look like this:

$$y_2 = k_2 x + d.$$

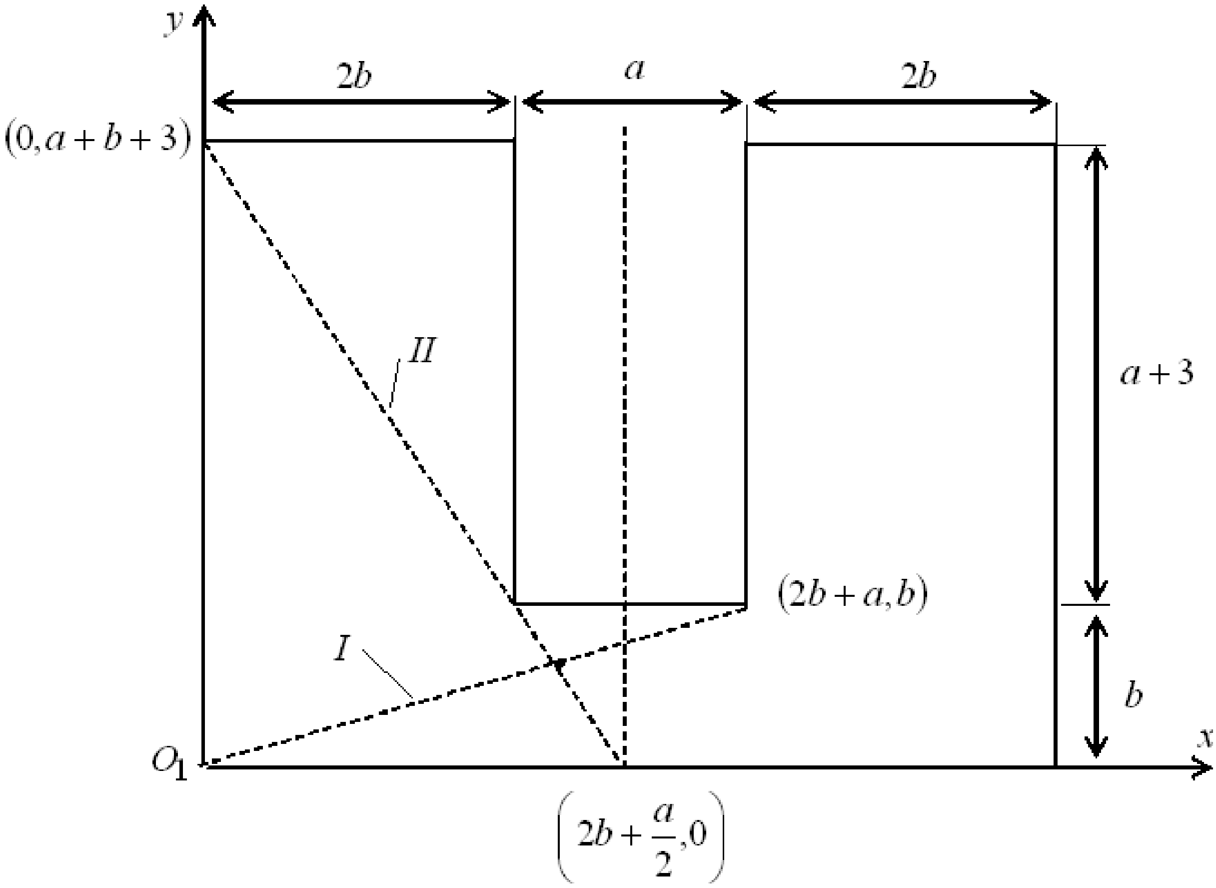

Fig. 27.7

Given that it passes through a point (from Figure 27.7), we get:

$$a + b + 3 = k_2 \cdot 0 + d\,,$$

where

$$d = a + b + 3.$$

Then the equation for the line II will look like:

$$y_2 = k_2 x + (a + b + 3).$$

This line still passes through the point $\left(2b + \frac{a}{2}, 0\right)$ (Fig. 27.7), but because:

$$0 = k_2\left(2b + \frac{a}{2}\right) + (a + b + 3),$$

where

$$k_2 = -\frac{a + b + 3}{2b + \frac{a}{2}} = -\frac{2a + 2b + 6}{4b + a}.$$

Consequently, the equation of the line II will have the form:

$$y_2 = -\frac{2a + 2b + 6}{4b + a}x + (a + b + 3).$$

To find the coordinates of the force application point, we need to solve a system that has the following form:

$$\begin{cases} y = \dfrac{b}{2b + a}x \\ y = -\dfrac{2a + 2b + 6}{4b + a}x + (a + b + 3) \end{cases}.$$

The solution of this system will look like:

$$x = \frac{(2b + a)(4b + a)(b + a + 3)}{8b^2 + 7ab + 2a^2 + 12b + 6a};$$

$$y = \frac{b(4b + a)(b + a + 3)}{8b^2 + 7ab + 2a^2 + 12b + 6a}.$$

Substituting the numerical values of the cross section parameters a and b, finally, we obtain the coordinates of the point of application of force F in the system O_1xy:

$$x_F = 4{,}597 \text{ dm}; \quad y_F = 1{,}313 \text{ dm}.$$

Let us determine the position of the centroid of a given cross-section, that is, the position of its central axes. Since the given cross-section has a vertical axis of symmetry, it will be one of its central axes. Therefore, we need to find only one coordinate of the centroid of the cross-section. It is y_C.

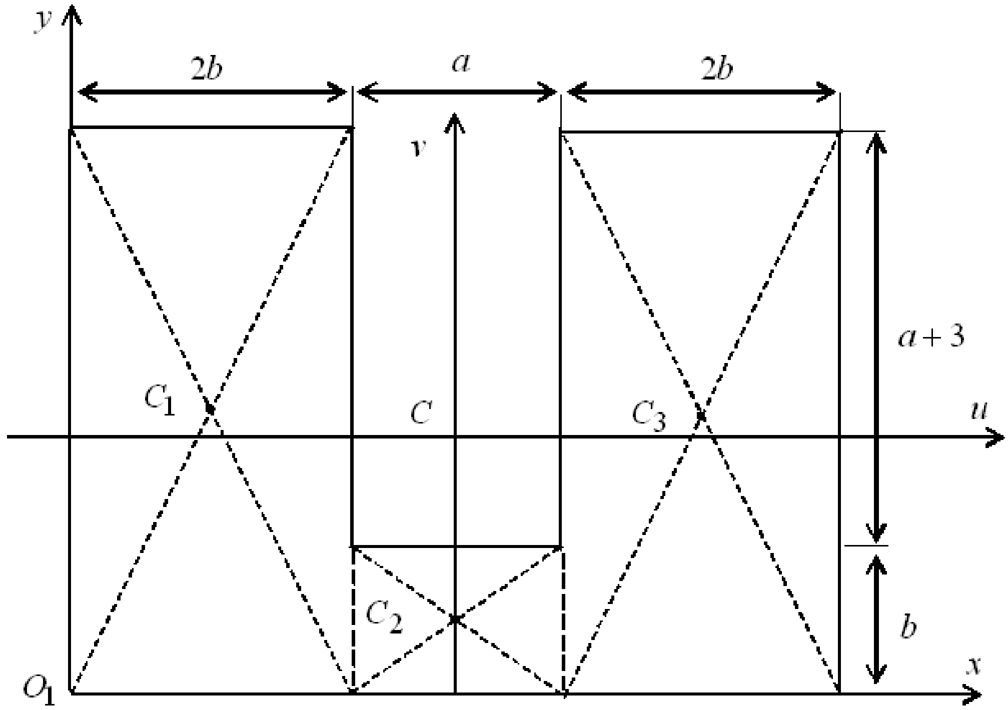

Fig. 27.6

Let us divide the conditionally given cross-section into three components according to Fig. 27.6. We compute the area of each of the constituent of cross-section and write the coordinates of their centres of weight relative to the axes x and y:

$$A_1 = 2b \cdot (b + a + 3) = 4 \cdot 8 = 32 \text{ dm}^2, \qquad y_{C_1} = \frac{b + a + 3}{2} = 4 \text{ dm};$$

$$A_2 = a \cdot b = 3 \cdot 2 = 6 \text{ dm}^2, \qquad y_{C_2} = \frac{b}{2} = 1 \text{ dm};$$

$$A_3 = A_2 = 32 \text{ dm}^2, \qquad y_{C_3} = \frac{b + a + 3}{2} = 4 \text{ dm}.$$

Then using the formula for centroid of complex cross-section, we obtain:

$$y_C = \frac{2A_1 \cdot y_{C_1} + A_2 \cdot y_{C_2}}{2A_1 + A_2} = \frac{2 \cdot 32 \cdot 4 + 6 \cdot 1}{2 \cdot 32 + 6} = \frac{262}{70} \approx 3{,}74 \text{ dm}.$$

We calculate the coordinates of the centres of weight for the components of cross-sections relative to the central axes u and v:

$$A_1 = 32\ \text{dm}^2, \qquad u_{C_1} = -3{,}5\ \text{dm}; \; v_{C_1} = 0{,}257\ \text{dm};$$
$$A_2 = 6\ \text{dm}^2, \qquad u_{C_2} = 0; \qquad v_{C_2} = -2{,}74\ \text{dm};$$
$$A_3 = 32\ \text{dm}^2, \qquad u_{C_3} = 3{,}5\ \text{dm}; \; v_{C_3} = 0{,}257\ \text{dm}.$$

We calculate the values of the central moments of inertia of a given section:

$$I_u = \left(I_{u_i} + v_{C_i}^2 \cdot A_i \right) =$$
$$= 2\left(\frac{4 \cdot 8^3}{12} + 0{,}257^2 \cdot 32 \right) + \left(\frac{3 \cdot 2^3}{12} + (-2{,}74)^2 \cdot 6 \right) = 392{,}705\ \text{dm}^4;$$
$$I_v = \left(I_{v_i} + u_{C_i}^2 \cdot A_i \right) =$$
$$= 2\left(\frac{8 \cdot 4^3}{12} + 3{,}5^2 \cdot 32 \right) + \left(\frac{2 \cdot 3^3}{12} + (0)^2 \cdot 6 \right) = 873{,}8\ \text{dm}^4.$$

We calculate the squares of the radii of inertia relative to the axes u and v:

$$i_u^2 = \frac{I_u}{A} = \frac{392{,}705}{70} = 5{,}61\ \text{dm}^2;$$
$$i_v^2 = \frac{I_v}{A} = \frac{873{,}8}{70} = 12{,}48\ \text{dm}^2.$$

Then we obtain:

$$u_F = -\frac{i_v^2}{a_u} = -\frac{12{,}48}{a_u};$$
$$\text{-}v_F = -\frac{i_u^2}{a_v} = -\frac{5{,}61}{a_v}.$$

Let us calculate the coordinates of the core of the cross-section, which is shown in Fig. 27.8:

1) $a_u = 5{,}5$ dm; then: $u_F = -\dfrac{12{,}48}{5{,}5} = -2{,}27$ dm

2) $a_v = 8 - y_C = 4{,}257$ dm; then: $v_F = -\frac{5{,}61}{4{,}257} = -1{,}32$ dm;

3) $a_u = -5{,}5$ dm; then: $u_F = -\frac{12{,}48}{-5{,}5} = 2{,}27$ dm;

4) $a_v = y_C \cdot (-1) = -3{,}74$ dm; then: $v_F = -\frac{5{,}61}{-3{,}74} = 1{,}5$ dm.

In the coordinate system O_1xy the point of application of the force F had coordinates:

$$x_F = 4{,}597 \text{ dm}; \quad y_F = 1{,}31 \text{ dm},$$

and in the coordinate system Ouv it has the following coordinates:

$$u_F = 4{,}597 - 5{,}5 = -0{,}903 \text{ dm};$$

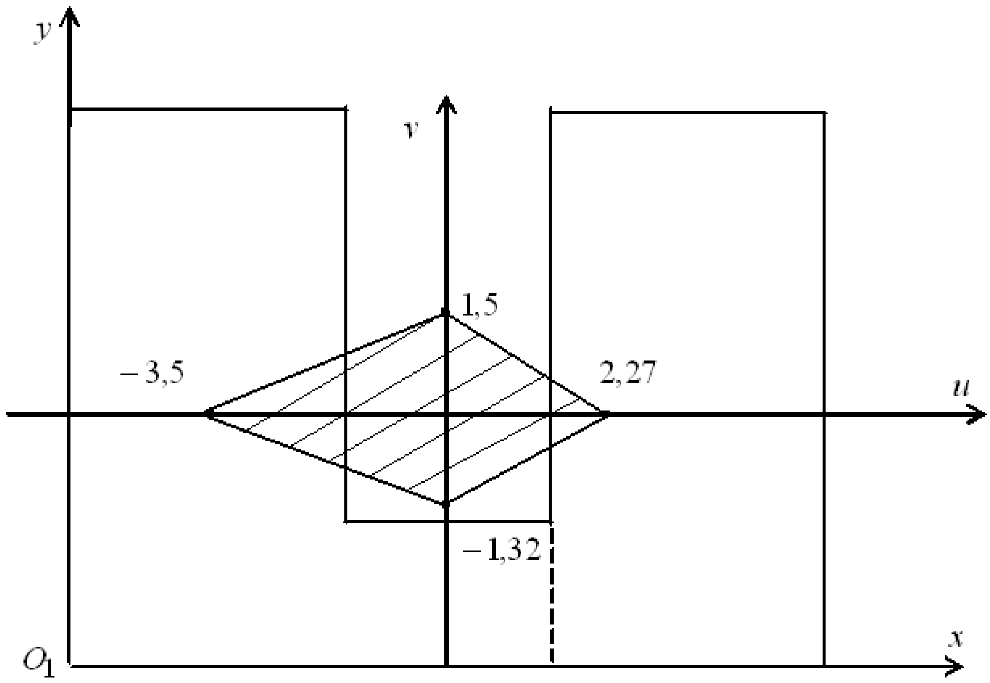

Fig. 27.8

$$v_F = 1{,}31 - 3{,}74 = -2{,}43 \text{ dm}.$$

This point does not get into the core of the cross-section, because the core of the cross-section has not the points with coordinates $v_F < -1{,}32$ dm.

On beginning