Lecture 29 - 30
COLUMNS

Plan

1. Definition of columns.
2. Critical load of a long slender column.
3. Typical example of calculation of column.

30.1 Definition of columns

A long slender bar subject to axial compression is called a **column**. The term „column" is frequently used to describe the vertical member, whereas the word „strut" is occasionally used in regard to inclined bars.

We can meet on practice many aircraft structural components, structural connections between stages of boosters for space vehicles, certain members in bridge trusses and structural frameworks of building are common examples of columns.

In practice, there are cases where external forces applied to the body (such as the bar) can be balanced by internal forces that occur in the body, deformed in several different forms. In this regard, two types of elastic equilibrium are distinguished. There are a stable and a unstable state of equilibrium. The unstable state of the column accompanies the buckling of column.

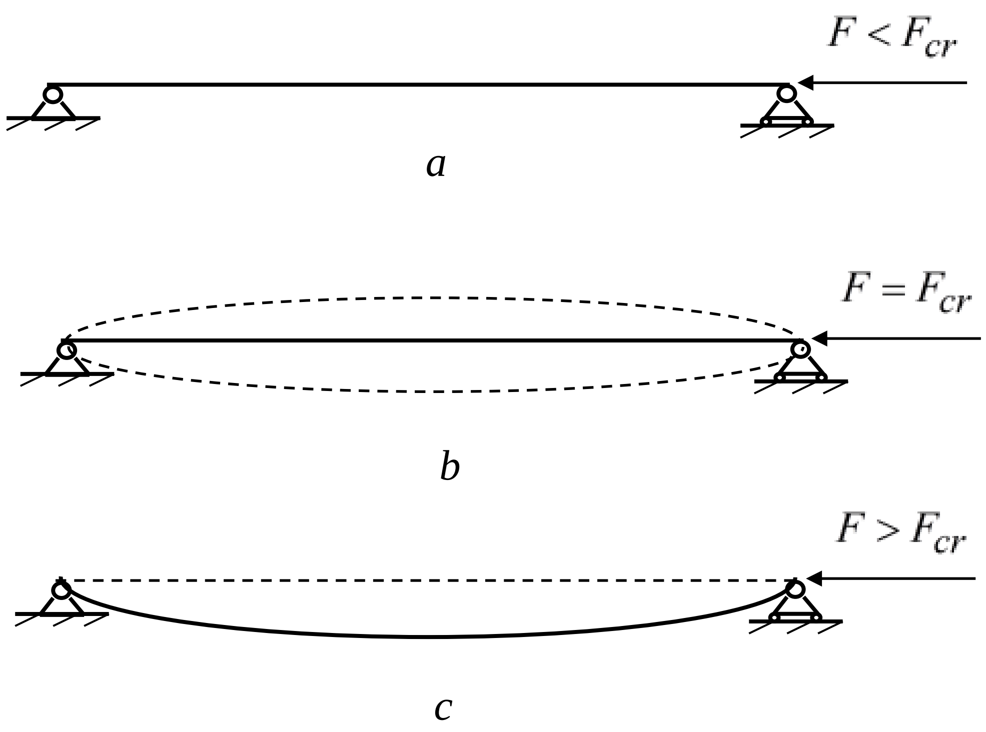

Fig. 30.1

If the deformed body at an arbitrary small deviation from the state of equilibrium tries to return to its initial state, as soon as the additional loads that caused the deviation are removed, then the elastic equilibrium is stable (see Fig. 30.1, b).

The case of equilibrium when the elastic body does not try to return to its original state after removing the additional (disturbing) forces that brought it from the initial equilibrium is called an unstable equilibrium. In the case of unstable equilibrium, the deformed elastic body continues to deform in the same direction in which it began to deform under the action of additional forces, even when these forces are removed (see Fig. 30.1, c).

The transition from a stable equilibrium to an unstable depends on the magnitude of the load. Between these two states of the body there is a transition state, which is called critical, and the magnitude of the load corresponding to this state is called a critical load. For example, if the bar is compressed by force, then the critical value of this force is noted as F_{cr}.

Since, the critical load of a slender bar subject to axial compression is that value of the axial force that is just sufficient to keep the bar in a slightly deflected configuration (Fig. 30.1, b).

If a compressive force F is applied to one end of the bar that can freely compress (and bend) in the direction of its longitudinal axis, and the second end is fixed rigidly (Fig. 30.1, a), then the bar is in a state of stable equilibrium until the value of the force F will be less than force F_{cr}. When the magnitude of force F becomes meaningful F_{cr}, the smallest increase in force can lead him out of this state and it suddenly buckling to one side or the other (Fig. 30.1, b). In fig. 30.1, c shows the bending of the bar under the action of force $F > F_{cr}$, however, it should be noted that in this case the value of force F, if the bar was still simply compressed (without bending), there would not be plastic deformation.

It is important to understand that the equilibrium form is not always changed when the value of force F_{cr} is reached. The value of force F can reach a larger value, and the bar will be simply compressed. However, at any moment, when $F \geq F_{cr}$, this change may suddenly occur.

On beginning

30.2. Critical load of a long slender column.

In practice, the achievement of critical load values is actually equal to the destruction of structures, because the unstable form of equilibrium will be lost due to unlimited growth of deformations and stresses. The threat of destruction of the structure due to loss of stability can be perceived as unexpected, as it comes suddenly at relatively low voltage values, when the strength of the structural element still has a large margin. If the load has not reached a critical value yet, the deformation by magnitude is negligible and almost not noticeable, but when the critical value reaches the point of destruction, the residual deformation is growing very quickly and actually there is not enough time to prevent the catastrophe.

Consequently, when calculating the stability of the critical load is similar to the destructive calculations for strength. To ensure the stability reserve, the condition is required:

$$F \le [F] = \frac{F_{cr}}{n_{st}}, \tag{30.1}$$

where F, which is acting on bar; n_{st} - coefficient of stability reserve.

From the above it follows the need to determine the critical loads.

Let us have a straight-line prismatic bar, which has a length of ℓ and bending stiffness EI (Fig. 30.2).

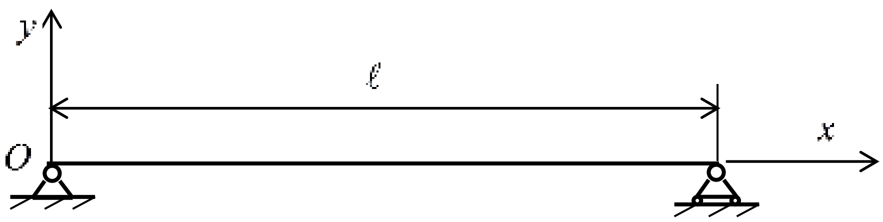

Fig.30.2

Let the longitudinal axis will be an axis Ox with a initially point O in accordance with Fig. 30.2. The left end of the bar is fixed by a fixed hinge, and the right is fixed by movable hinge. Let us assume that the compressive force F is bigger than the force F_{cr} on a very small value.

To determine the critical force, we first write the differential equation of the elastic line:

$$EI\frac{d^2y}{dx^2} = M(x). \tag{30.2}$$

The distorted bar axis is shown in fig. 30.3.

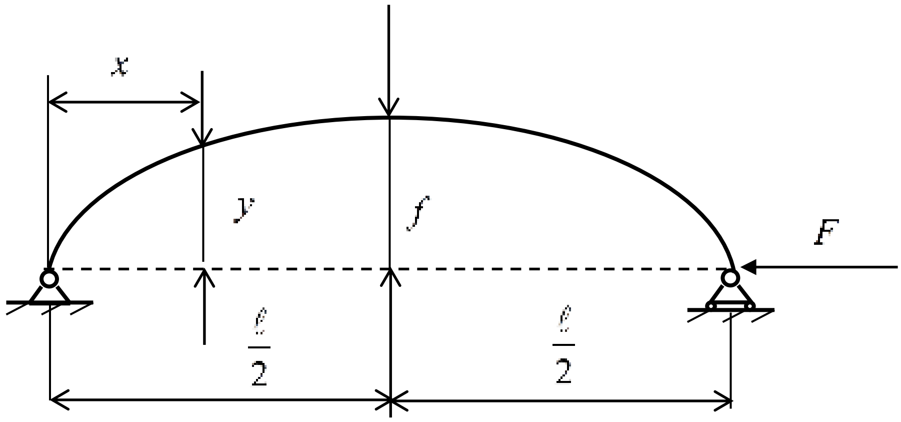

Fig. 30.3

In this case, let us take into account that the deflection is perpendicular to the longitudinal axis with the least rigidity of the cross-section of the bar. Because:

$$|M(x)| = |Fy|,$$

and the sign of the second derivative is opposite to the deflection sign y, then it is clear that equation (30.2) is valid when the bar is bending both up and downward (in the direction y).

So, we have:

$$EI_{\min} \frac{d^2 y}{dx^2} = -Fy. \tag{30.3}$$

For convenience, the equation (30.3) is rewritten in the following form:

$$\frac{d^2 y}{dx^2} + k^2 y = 0, \tag{30.4}$$

where the notation is entered:

$$k^2 = \frac{F}{EI_{\min}}. \tag{30.5}$$

The general solution of the second-order homogeneous linear equation (30.4) is written, as we know, as follows:

$$y(x) = A\sin kx + B\cos kx. \tag{30.6}$$

To determine the constant of integration A and B the boundary conditions have to be fulfilled:

$$y(x)\big|_{x=0} = 0, \quad y(x)\big|_{x=\ell} = 0. \qquad (30.7)$$

From the first boundary condition (30.7), we obtain that $B \equiv 0$.

Then the solution (30.6) will look like:

$$y(x) = A\sin kx. \qquad (30.8)$$

Follow the second condition (30.7):

$$A\sin k\ell = 0,$$

since the trivial variant $A = 0$ does not have physical content, we will obtain:

$$\sin k\ell = 0. \qquad (30.9)$$

From equation (30.9) we obtain an infinite set of roots:

$$k\ell = n\pi, \quad n = 1,2,3,\ldots. \qquad (30.10)$$

The root $k\ell = 0$ does not match the initial data and we reject it as unnecessary.

Then the equation (30.10) will be rewritten as follows:

$$k^2\ell^2 = n^2\pi^2, \quad n = 1,2,3,\ldots. \qquad (30.11)$$

Taking into account the notation (30.5) from equation (30.11) follows that:

$$F_n = \frac{\pi^2 EI_{\min}}{\ell^2} n^2. \qquad (30.12)$$

The formula (30.12) is called the Euler formula.

The most practical value is the smallest value of F_{cr}, which is obtained from formula (30.12) with $n = 1$.

$$F_{cr} = \frac{\pi^2 EI_{\min}}{\ell^2}. \tag{30.13}$$

This formula was first obtained by the Swiss mathematician Leonard Euler and the load F_{cr} is called the Euler buckling load.

Then equation (30.8) takes the form:

$$y(x) = A \sin \frac{n\pi x}{\ell}. \tag{30.14}$$

When $\sin \frac{n\pi x}{\ell} = 1$ we get the biggest deflection of bar, which is $f = A$. The equation of the elastic line takes the form:

$$y = f \sin \frac{n\pi x}{\ell}. \tag{30.15}$$

As follows from equality (30.15), the deflection $y = 0$ will be at points:

$$x = \frac{\ell}{n};\ \frac{2\ell}{n};\ \frac{3\ell}{n};\ \ldots;\ \frac{m\ell}{n}, \quad m = 1,2,3,\ldots,n, \tag{30.16}$$

and in the middle between two neighboring points the deflection will be modulo the largest $|y|_{\max} = f$.

It is clear that the shape of the bend of the bar has a half-wave length of the sinusoid at equal length ℓ is equal n. At the $n = 1$ the bar has the one half-wave of sinusoid.

When we have derived Euler's formula, the bar was simply supported. This case of supporting is considered to be the main one. It is necessary to consider other cases of fastening the bar, which is known to meet other boundary conditions.

Formula for the smallest critical compressive force value (30.13), can be combined rewrite in general form:

$$F_{cr} = \frac{\pi^2 EI_{\min}}{(\nu\ell)^2}, \tag{30.44}$$

where $\nu\ell = \ell_{el}$ - an effective length of the column, ℓ - actual length of the bar, ν - coefficient of lengthening.

Consequently, different cases of supports and load of the bar are given to the main case by introducing into the formula for Euler buckling load F_{cr} the so-called effective length of the column $\ell_{el} = \nu\ell$:

- $\nu = 1$ - for a column pinned at both ends;
- $\nu = 0{,}7$ - for a column, which one end clamped and the other pinned;
- $\nu = 0{,}5$ - for a column, which both ends are rigidly clamped;
- $\nu = 2$ - in the case of a cantilever-type column loaded at its free end.

Previously, the gates of the rod's stability were considered, provided Hooke's law took place. This means that the stresses calculated on the basis of the found critical force:

$$\sigma_{cr} = \frac{F_{cr}}{A} \le \sigma_{yp}. \tag{30.56}$$

They did not exceed the limits of proportionality.

If this condition is not satisfied, then the differential equations (30.2) can not be used.

Then the formula for critical stresses σ_{cr}, proceeding from the formulas (30.44) and (30.56) is obtained:

$$\sigma_{cr} = \frac{F_{cr}}{A} = \frac{\pi^2 E I_{\min}}{A(\nu\ell)^2} = \frac{\pi^2 E}{\left(\frac{\nu\ell}{i}\right)^2} = \frac{\pi^2 E}{(\lambda)^2}, \tag{30.57}$$

where $i^2 = i_{\min}^2 = \frac{I_{\min}}{A}$ - the square of the smallest of the main radius of gyration of the cross-section of the; $\lambda = \frac{\nu\ell}{i}$ - slenderness ration of the column.

Thus, the critical stress depends only on the elastic characteristic - the modulus of elasticity E and flexibility of the bar λ.

Dependence (30.57) is a transformed Euler formula.

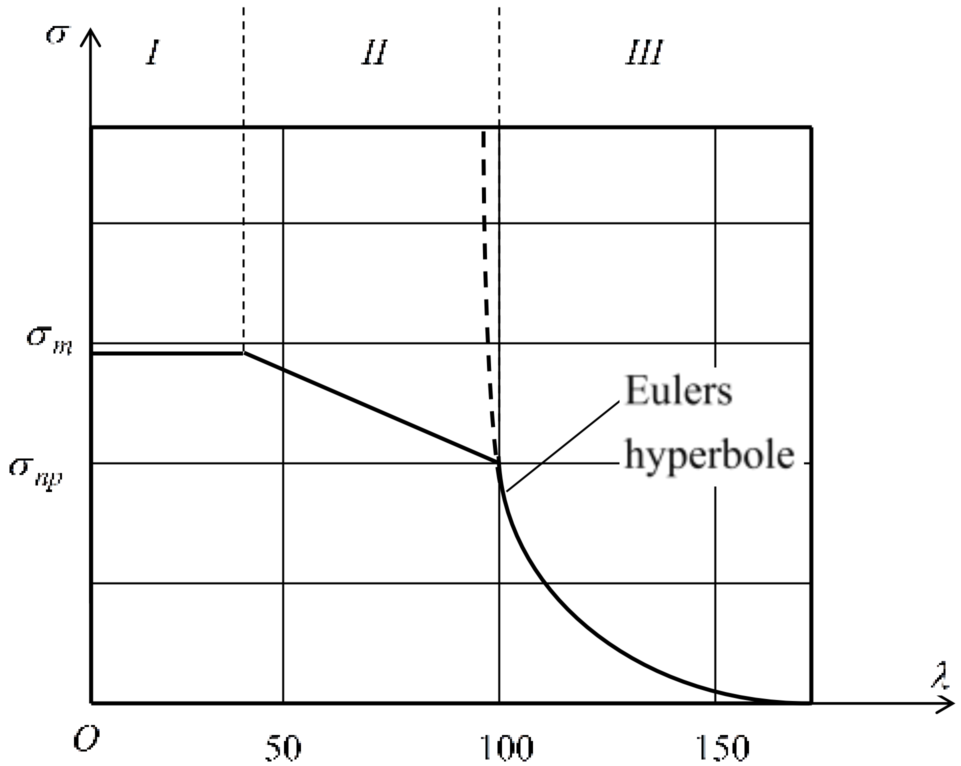

Fig. 30.4

If we introduce the coordinate system $\sigma_{cr} - \lambda$ (Fig. 30.4), then the dependence (30.57) will be given as hyperbolic (Euler). At the same time, when slenderness ration of the column increases, the critical stress will go to zero, and when the slenderness ration of the column goes to zero, then the critical value of the stresses will increase indefinitely, which even can be seen directly from formula (30.57).

However, from condition (30.56):

$$\sigma_{cr} = \frac{\pi^2 E}{\lambda^2} \le \sigma_{yp},$$

from formula (30.57) follows:

$$\lambda_{\lim} \ge \sqrt{\frac{\pi^2 E}{\sigma_p}}. \tag{30.58}$$

Consequently, the Euler formula can not be used with flexibility less than the limit value of it ($\lambda_{\lim}$).

Solving the problem of stability beyond the limits of proportionality is associated with great difficulty, and, in theory, there is no solution.

Some scientists have established empirical formulas, processing a large number of experimental data.

So F.S. Yasinsky wrote the following formula:

$$\sigma_{cr} = a - b\lambda. \tag{30.59}$$

The formula (30.59) is suitable for constructing a plot in the second section for plastic materials, which is shown on Fig. 30.4. For fragile materials, such as cast iron, they use parabolic dependence:

$$\sigma_{cr} = a - b\lambda + c\lambda^2, \tag{30.60}$$

taking that $c = 0{,}53$.

The values of the coefficients a and b for the investigated materials are given in Table 30.1.

Table 30.1

Material	$\lambda_{\lim}$	a	b
		MPa	
St. 2, St. 3	100	310	1,14
St. 5	100	464	3,26
Steel 40	90	321	1,16
Cast iron	80	776	12
Tree (pine)	110	29,3	0,194

The data presented in Table 30.1 together with the formulas (30.59) and (30.60) provide an opportunity for each material when $0 < \lambda < \lambda_{\lim}$ constructing diagram of the dependence of critical stresses on the flexibility of the bar.

The value of critical stresses σ_{cr} are calculated by the formulas (30.59) and (30.60), at a certain value of flexibility $\lambda = \lambda_0$, becomes equal to the boundary stress during compression for plastic materials:

$$\sigma_{cr} = \sigma_{pr}, \tag{30.61}$$

and for fragile materials:

$$\sigma_{cr} = \sigma_e. \tag{30.62}$$

If we have condition, that $\lambda < \lambda_0$, then the bar is called the bar of low flexibility and it is calculated only for strength.

The calculation of the bar for stability is based on the coefficient of reduction of the basic admissible stress φ, i.e.:

$$[\sigma]_{st} \le \varphi[\sigma], \tag{30.63}$$

where $[\sigma]_{st} = \dfrac{\sigma_{cr}}{n_{st}}$ - admissible stress on stability,

$[\sigma] = \dfrac{\sigma_{yp}}{n_{yp}}$ - admissible stress on compressive strength.

The stability condition for the compressed bars has the form:

$$\sigma \le [\sigma]_{st}, \tag{30.64}$$

or taking into account (30.63):

$$\sigma = \frac{F}{A} \le \varphi[\sigma]. \tag{30.65}$$

On beginning

30.3. Typical example of calculation of column

Let the bar, which is depicted in Fig. 30.5, a, has a length of 9 m. The compression force of 54 N. is applied to the upper end of the bar.

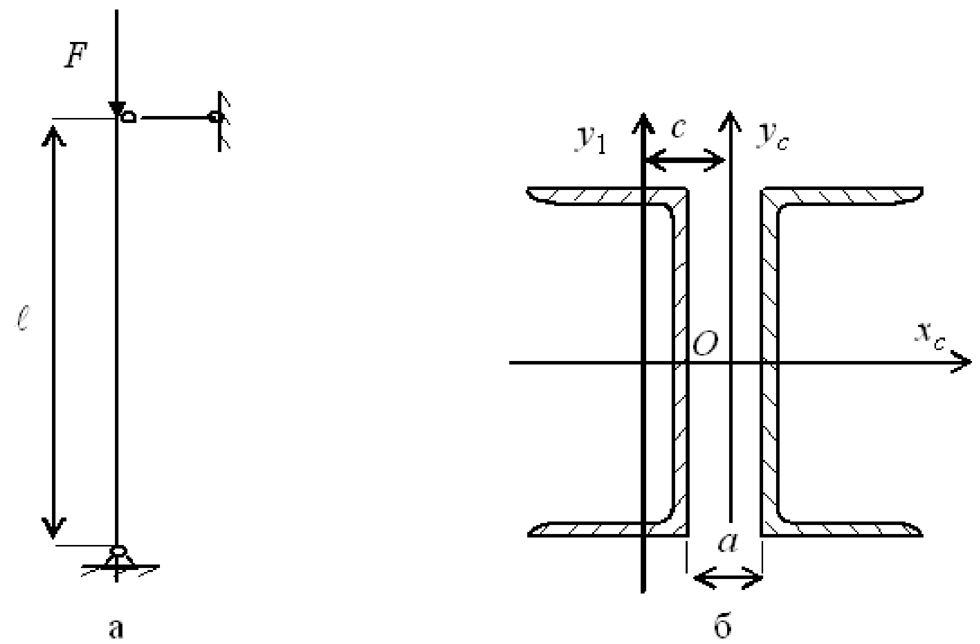

Fig. 30.5

Pick up the number of channels, of which the cross section is consisted (Fig. 30.5, b).

From the condition of stability to determine the distance a.

Select the number of the channels, from which the cross section is formed (Fig. 30.5, b).

I. approximation.

Let us assume that the column is made of steel St3.

Then, for the first approximation, we choose the ratio φ_1 as the arithmetic mean of the largest and smallest of its values:

$$\varphi_1 = \frac{0,19+1}{2} \approx 0,6.$$

Then

$$[\sigma]_{st} = [\sigma]\cdot\varphi_1 = 16\cdot 0,6 = 9,6 \text{ kPa}.$$

Determine the area of the channel:

$$A_c \geq \frac{F}{2[\sigma]_{st}} = \frac{54}{2\cdot 9,6\cdot 10^3} = 2,81\cdot 10^{-3} \text{ m}^2 = 28,1 \text{ cm}^2.$$

From the table of steel assortment we choose the channel № 22 a, which has $A = 28,8 \text{ cm}^2$ and $I_x = 2330 \text{ cm}^4$.

Find the radius of gyration of the cross section:

$$i_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{2330}{28,8}} \approx 9 \text{ cm}.$$

Calculate the slanders of column:

$$\lambda = \frac{\nu\ell}{i_x} = \frac{0,7\cdot 900}{9} = 70.$$

Determine the value of the coefficient φ from the corresponding table:

$$\varphi_1' = 0,81.$$

So, you need to make a second approximation.

II. approximation.

$$\varphi_2 = \frac{0{,}81+0{,}6}{2} \approx 0{,}705.$$

Then:

$$[\sigma]_{st} = [\sigma]\cdot\varphi_1 = 16\cdot 0{,}705 = 11{,}28 \text{ kPa}.$$

Determine the area of the channel:

$$A_c \geq \frac{F}{2[\sigma]_{st}} = \frac{54}{2\cdot 11{,}28\cdot 10^3} = 2{,}39\cdot 10^{-3}\ \text{m}^2 = 23{,}9\ \text{cm}^2.$$

From the table of steel assortment we select the channel number 20a, which has $A = 25{,}2\ \text{cm}^2$ and $I_x = 1670\ \text{cm}^4$.

Find the radius of gyration of the cross section:

$$i_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{1670}{25{,}2}} \approx 8{,}1\ \text{cm}.$$

Calculate the slanders of column:

$$\lambda = \frac{\nu\ell}{i_x} = \frac{0{,}7\cdot 900}{8{,}1} = 77{,}8.$$

Determine the value of the coefficient φ from the corresponding table. Since the found value of the slanders of the bar λ is in the range of 70 to 80, then considering that $\lambda = 70$ when, $\varphi = 0{,}81$, and with $\lambda = 80$ $\varphi = 0{,}75$, we obtain:

$$\varphi_2' = 0{,}81 - 0{,}006\cdot 7{,}8 \approx 0{,}763.$$

$$\Delta = \frac{0{,}763-0{,}705}{0{,}705}\cdot 100\ \% = 8{,}2\ \% > 5\ \%.$$

So, you need to make a third approximation.

III approximation.

$$\varphi_3 = \frac{0{,}705 + 0{,}763}{2} \approx 0{,}734.$$

Then:

$$[\sigma]_{st} = [\sigma] \cdot \varphi_1 = 16 \cdot 0{,}734 = 11{,}74 \text{ MPa}.$$

Determine the area of the channel:

$$A_c \geq \frac{F}{2[\sigma]_{st}} = \frac{54}{2 \cdot 11{,}74 \cdot 10^3} = 2{,}3 \cdot 10^{-3} \text{ m}^2 = 23 \text{ cm}^2.$$

From the table of steel assortment we choose a channel number 20, which has $A = 23{,}4$ cm^2 and $I_x = 1520$ cm^4.

Find the radius of gyration of the cross section:

$$i_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{1520}{23{,}4}} \approx 8{,}06 \text{ cm}.$$

Then the slanders of column will be equal:

$$\lambda = \frac{\nu \ell}{i_x} = \frac{0{,}7 \cdot 900}{8{,}06} = 78{,}2.$$

Determine the value of the coefficient φ from the corresponding table. The newly found value of the bar's flexibility λ is in the range of 70 to 80, then we obtain:

$$\varphi_2' = 0{,}81 - 0{,}006 \cdot 8{,}2 \approx 0{,}761.$$

$$\Delta = \frac{0{,}761 - 0{,}734}{0{,}734} \cdot 100 \text{ \%} = 3{,}6 \text{ \%} < 5 \text{ \%}.$$

So let us stop at the №. 20 channel.

On condition of stability, this has following form:

$$I_{x_c} = I_{y_c},$$

determine the distance c (Fig. 30.5, b):

$$2 \cdot 1520 = 2\left(113 + c^2 \cdot 23{,}4\right),$$

where

$$c = \sqrt{\frac{1520 - 113}{23{,}4}} = 7{,}75 \text{ cm.}$$

Then the desired distance a will be equal:

$$a/2 = c - z_0,$$

$$a = 2\left(c - z_0\right) = 2\left(7{,}75 - 2{,}07\right) = 11{,}36 \text{ cm.}$$

On beginning